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ALGEBRA

Combinatorial Analysis

Abrham, Jaromír; und Driml, Miloslav. Über ein Problem der Kodentheorie. Časopis Pěst. Mat. 81 (1956), 69–76. (Czech. Russian and German summaries)

Given five alphabets of n letters each, the authors describe a method of constructing, for each triple of parameters $0 \leqq a, b, c < n$, a set of n^2 words $[\alpha_1, \cdots, \alpha_5]$ such that α_i belongs to the ith alphabet. Any set constructed by their method has the property (P) that any two members differ in at least three places. They then give necessary and sufficient conditions on two triples, under which the union of the word sets generated by each triple also has (P). Finally they show non-constructively that there exists a choice of n distinct triples such that the union of the sets of words generated by each triple still has (P). This choice generates the largest possible set having P. *V. E. Beneš* (Murray Hill, N.J.).

See also: Hall and Swift, p. 192; Cotlar, p. 219; Roy, p. 254; Archbold and Johnson, p. 244; Sprott, p. 244.

Elementary Algebra

★**Alexandroff, P. S.; Markuschewitsch, A. I.; and Chintschin, A. J. Enzyklopädie der Elementarmathematik. Band II, Algebra.** VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. ix+405 pp.

A translation of Ènciklopediya èlementarnoĭ matematiki, v. 2 [Gostehizdat, Moscow-Leningrad, 1951].

★**Levi, Howard. Elements of algebra. Second edition.** Chelsea Publishing Co., New York, 1956. viii+160 pp. $3.25.

The first edition was reviewed in MR 16, 325.

Gonçalves, J. Vicente. Sur la décomposition des fractions rationnelles. Univ. Lisboa. Revista Fac. Ci. A. (2) 5 (1955–1956), 171–176.

This is an exposition of a method for the decomposition of a rational fraction [cf. Gonçalves, Curso de algebra superior, v. 1, 2d ed., Lisbon, 1944]. *E. Frank.*

Linear Algebra

Perfect, Hazel. A lower bound for the diagonal elements of a non-negative matrix. J. London Math. Soc. 31 (1956), 491–493.

It is known that the diagonal elements χ of a non-negative matrix do not exceed the dominant eigenvalue λ_1. Here it is shown that they have the lower bound

$$\Lambda = \frac{1}{n}\{\sum \lambda_i - [(n-1)\sum_{i<} (\lambda_i - \lambda_j)^2]^{1/2}\},$$

where λ_i are the eigenvalues and n the dimension of the matrix. The author suggests the problem of finding conditions on λ_i such that the inequalities $\lambda_1 \geqq \chi \geqq \max(0, \Lambda)$ are sufficient for the existence of a non-negative matrix with a given χ in the diagonal. If the dominant root is not simple then the conditions are in general not sufficient. On the other hand they are sufficient if the $\lambda_i \neq \lambda_1$ all coincide. *O. Taussky-Todd* (Washington, D.C.).

Fan, Ky. A comparison theorem for eigenvalues of normal matrices. Pacific J. Math. 5 (1955), 911–913.

The author proves the following result. Let M and N be normal $n \times n$ matrices over the complex field, and let $\mu_1^2, \mu_2^2, \cdots, \mu_n^2$ be the eigenvalue of $(M-N)^*(M-N)$, arranged in descending order of magnitude. Let $\varepsilon \geqq \mu_{r+1} \geqq 0$. If the closed circular disc $|z-z_0| \leqq \rho$ contains p eigenvalues of M, then the concentric disc $|z-z_0| \leqq \rho+\varepsilon$ contains at least $p-r$ eigenvalues of N.

The following theorem (oral communication attributed to H. Wielandt) is obtained as a corollary. If M and N are normal $n \times n$ matrices, $M-N$ has rank r, and the closed circular disc D in the complex plane contains exactly p eigenvalues of M and q eigenvalues of N, then $|p-q| \leqq r$. *F. Smithies* (Cambridge, England).

de Bruijn, N. G. Inequalities concerning minors and eigenvalues. Nieuw Arch. Wisk. (3) 4 (1956), 18–35.

The author presents a systematic treatment of inequalities concerning the minors and eigenvalues of complex matrices. The principal tools used are the Binet-Cauchy theorem on compound matrices, the Laplace expansion theorem for determinants and the Cauchy-Schwarz inequality.

Let Greek letters $\sigma, \tau, \cdots$ denote sets of k numbers drawn from $1, 2, \cdots, n$, and let $A_{\sigma\tau}$ denote the k-rowed matrix derived from the $n \times n$ matrix A by deleting all rows whose indices do not belong to σ and all columns whose indices do not belong to τ. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A, arranged so that $|\lambda_1| \geqq |\lambda_2| \geqq \cdots \geqq |\lambda_n|$. Write

$$g_k(A) = \max_{\sigma,\tau} |\det A_{\sigma\tau}|, \quad f_k(A) = \max_{\sigma} |\det A_{\sigma\sigma}|,$$

$$\varphi_k(A) = |\lambda_1\lambda_2 \cdots \lambda_k|, \quad p_k(A) = \sum_{\sigma} \det A_{\sigma\sigma}.$$

The main results proved are as follows:

$$g_k^2(A_1A_2 \cdots A_m) \leqq f_k(A_1A_1^*) \cdot \prod_{i=2}^{m-1} \varphi_k(A_iA_i^*) \cdot f_k(A_m^*A_m) \quad (m \geqq 2),$$

$$\varphi_k^2(A_1A_2 \cdots A_m) \leqq \prod_{i=1}^{m} \varphi_k(A_iA_i^*) \quad (m \geqq 1),$$

$$g_k^2(A_1A_2 \cdots A_m) \leqq \prod_{i=2}^{m} \varphi_k(A_iA_i^*) \quad (m \geqq 1);$$

if $x_1, \cdots, x_k, y_1, \cdots, y_k$ are vectors,

$$|\det [x_i^* A y_j]|^2 \leq \varphi_k(AA^*) \det [x_i^* x_j] \det [y_i^* y_j];$$

if $h \leq k$,

$$\varphi_h^2(A_{\sigma\sigma}) \leq \varphi_h(AA^*); \quad |p_k(AB)|^2 \leq p_k(AA^*) p_k(BB^*);$$

and, if σ' is the complementary set of σ,

$$|\det A|^2 \leq \det (AA^*)_{\sigma\sigma} \det (AA^*)_{\sigma'\sigma'}.$$

Extensions to rectangular matrices and applications to R. Courant's minimax principle are also given.

Numerous known results are derived from the above inequalities. Among these may be mentioned:

$$\varphi_k^2(A) \leq \varphi_k(AA^*)$$

[H. Weyl, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 408–411; MR **11**, 37; also given, as noted in a supplementary communication from the author, in H. W. Turnbull and A. C. Aitken, An introduction to the theory of canonical matrices, Blackie, London, 1932, p. 110, Ex. 17];

$$\varphi_k(ABB^*A^*) \leq \varphi_k(AA^*) \varphi_k(BB^*)$$

[A. Horn, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 374–375; MR 13, 565];

$$\varphi_1^2(AB) \leq \varphi_1(AA^*) \varphi_1(BB^*)$$

[S. N. Roy, Proc. Amer. Math. Soc. **5** (1954), 635–638; MR **16**, 4]; and the well-known determinant inequality

$$|\det A|^2 \leq \prod_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2$$

of J. Hadamard. *F. Smithies* (Cambridge, England).

★ **van Albada, Pieter Jacob. Integral relations in alternative coordinate rings.** Thesis, Rijksuniversiteit te Utrecht, 1955. 44 pp.

The author considers alternative polynomials $P = P(x_1, \cdots, x_n)$ with rational integral coefficients; that is, elements of the free alternative ring F generated by the rational integers and a set of indeterminates $x_1, \cdots, x_n$. If R is an alternative ring possessing a unit element and a characteristic k (zero or prime), the set A of all polynomials of F which yield identical relations for R is an ideal with the following properties: (1) If $P(x_1, \cdots, x_n)$ is in A and $y_1, \cdots, y_n$ are in F, then $P(y_1, \cdots, y_n)$ is in A. (2) If m is a positive integer and if either $k=0$ or $(m, k)=1$, then P is in A if mP is. In case R has no (nonzero) nilpotent elements, A also satisfies (3) P is in A if some positive power of P is in A. An ideal of F is called a T-ideal (corresponding to characteristic k) if it satisfies (1), (2); and a TR-ideal, if it satisfies (1), (2), (3). A polynomial P is called T-nuclear (TR-nuclear) if the T-ideal (TR-ideal) generated by P contains no nontrivial polynomial which is "simpler" than P in a sense made precise by the author.

In terms of the commutator $[x, y] = xy - yx$, the associator $(x, y, z) = (xy)z - x(yz)$ and the extended commutators $[x_1, \cdots, x_{n+1}] = [[x_1, \cdots, x_n], x_{n+1}]$, the following polynomials are shown to be T-nuclear for $k \neq 2$: $[b, a]$, $[b, a, a]$, (a, b, c), $[b, a, a, a]$, $[b, a]^2$, $[a, (a, b, c)]$, $[[a, b], [a, c]]$ and $\{a, b|c, d\} = h(a, b, c, d) + h(a, c, d, b) + h(a, d, b, c)$ where $h(a, b, c, d) = [a, b]\,[c, d] + [c, d]\,[a, b]$. Other identity polynomials are isolated but not shown to be T-nuclear. Further results for $k=2$ and for five indeterminates with $k \neq 2$ are derived on the assumption of the associative law. In connection with proofs of T-nuclearity the author defines various alternative rings, in particular one consisting of certain matrices of order 4 with elemenfs in an alternative ring. The methods are akin to those to Specht [Math. Z. **52** (1950), 557–589; MR **11**, 711].

Turning to TR-nuclear polynomials, the author considers an alternative ring R with a unit, without nonzero nilpotent elements, and of characteristic k (zero or prime). He proves the following: (I) If R satisfies $[b, a, a, a, a]=0$, then R is commutative. (II) If $k \neq 2$, 3 and R satisfies $[a, (a, b, c)]=0$, then R is associative. (III) If $k \neq 2$ and R is a division ring satisfying $[[a, b]^2, a]=0$, then R is a field, a quaternion algebra or a Cayley algebra. He also studies certain non-integral identical relations (involving inverses of elements) which require too much space to be exhibited here. For (I) the author uses the methods of Dehn [Math. Ann. **85** (1922), 184–192]; for (II) he requires those of Bruck and Kleinfeld [Proc. Amer. Math. Soc. **2** (1951), 878–890; MR **13**, 526]. He shows that (III) follows rapidly from results of Amitsur and Levitzki [ibid. **2** (1951), 320–327; MR **12**, 669] and Kaplansky [Bull. Amer. Math. Soc. **54** (1948), 575–580; MR **10**, 7] but also gives a longer independent proof.

{Reviewer's comments. The author went to Indonesia in 1951 and appears to have had little access to mathematical literature since that time. For this reason, and with no thought of depreciating the paper, I add some clearly relevant references: (i) Smiley [Ann. of Math. (2) **49** (1948), 702–709; MR **10**, 6] proves the identity $(a, b, c)^3=0$ in every commutative alternative ring; this bears on (I) above. (ii) Kleinfeld [Proc. Amer. Math. Soc. **3** (1952), 348–351; MR **14**, 129] proves the identity $[(a, b, c)^2, a]^3=0$ in every alternative ring. (iii) Kleinfeld [Ann. of Math. **58** (1953), 544–547; MR **15**, 392] proves the identity $([a, b]^4, c, d)=0$ and gives enough to prove the identity $([a, b]^2, c, d)^3=0$ in every alternative ring. (iv) Kleinfeld [Amer. J. Math. **77** (1955), 725–730; MR **17**, 231] proves the identities $[(a, b, c)^2, d] = ((a, b, c)^2, d, e) = ([a, b]^2, c, d) = 0$ in every semi-simple alternative ring.}
R. H. Bruck (Madison, Wis.).

★ **Papy, Georges. Une propriété arithmétique des algèbres de Grassmann.** III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 40–45. Fédération belge des Sociétés Scientifiques, Bruxelles.

Let F be an exterior form in the Grassman algebra G, and $F^{\wedge k}$ the kth exterior power of F. The expression $1/k!\ F^{\wedge k} = F^k$ can be so written that it continues to have meaning when the coefficient ring R over G is an integral domain with characteristic 0 or p, even if p/k. Let $H = x_1 \wedge y_1 + \cdots + x_n \wedge y_n$, where $x_1, \cdots, x_n, y_1, \cdots, y_n$ is a basis of order $2n$ in G. The principal theorem asserts that if F has a degree $d \geqq n+k$, then F is divisible by H^k if and only if $(k+i)_k$ divides $F \wedge H^i$ for $i = 1, 2, \cdots, [n-d/2]$, where $(k+i)_k$ is the number of combinations of $k+i$ things taken k at a time and $[x]$ is the greatest integer $\leq x$. In case the integral domain R is a field of characteristic zero the results degenerate into a theorem of Lepage [Bull. Soc. Roy. Sc. Liège **15** (1946), 21–31; MR **8**, 499].
A. N. Milgram (Minneapolis, Minn.).

Matschinski, M. Über eine Form der Lösung linearer Gleichungsysteme. Portugal. Math. **14** (1956), 133–139.

It is well-known that with the minimum of a quadratic form in n variables, is connected a system of n linear simultaneous equations in the n unknowns. The corresponding matrix is symmetric. The author is concerned with the solution of such a system, particularly when the

equations have been reduced to the triangular form. This reduction itself involves the calculation of the inverse of the triangular matrix after which the problem offers no difficulty. In fact, the matrix $[C_{ji}/C_{jj}c_{jj}]$ is, in the notation of the author, the inverse of $[c_{ij}]$. *H. Gupta.*

See also: Remež, p. 206; Łukaszewicz and Warmus, p. 235; Bodewig, p. 235.

Polynomials

Krishnaiah, P. V.; and Subrahmanyam, N. V. On elementary symmetric functions. J. London Math. Soc. 31 (1956), 364–369.

It is known that if a functional relation $\varphi(u_1, u_2, u_3)=0$ holds between the elementary symmetric functions

$$u_1=u_1(x_1, x_2, x_3) = f(x_1)+f(x_2)+f(x)_3,$$
$$u_2=f(x_2)f(x_3)+f(x_3)f(x_1)+f(x_1)f(x)_2,$$
$$u_3f(x_1)f(x_2)f(x_3),$$

then f is constant. In this result it is tacitly assumed that f and φ are real valued and differentiable in the relevant neighbourhoods. The author raises the question of necessity of the differentiability conditions, and of the possibility of extending the result to complex valued functions.

$u_r(t_1, t_2, \cdots, t_n)$, $r=1, 2, \cdots, n$, is a normal set of functions over an open n-dimensional set 0 if the u_r's are differentiable over 0; and if (1) the set at which, for some β, $\prod_{\alpha=1}^{n} J_{\alpha\beta}\neq 0$ is everywhere dense in 0, $J_{\alpha\beta}$ is the minor $(\partial u/\partial\alpha)/(\partial t/\partial\beta)$ in the Jacobian

$$D(u_1, u_2, \cdots, u_n)/(Dt_1, t_2, \cdots, t_n);$$

(2) the system of equations $u_r(t_1, t_2, \cdots, t_n)=a_r$ ($r=1, 2, \cdots, n$) has at most a denumerable set of solutions. A quasi-functional relation exists among the functions $v_r(x_1, x_2, \cdots, x_n)$ if, whenever the equations $v_r=a_r$, ($r=1, 2, \cdots, n$) are consistent in $x_1, x_2, \cdots, x_n$ there corresponds to $a_1, a_2, \cdots, a_n$ an index λ such that the set of values of v_λ consistent with the $n-1$ equations $v_r=a_r$, $r\neq\lambda$, is at most enumerable. Given n functions of v_r of the n variables $x_1, x_2, \cdots, x_n$ then v_λ is a quasi-function of the remaining $n-1$ functions if the set of values of v_λ consistent with any assigned values for v_r, $r\neq\lambda$, is at most enumerable.

It is shown that the theorem first stated holds when (A) f is real valued over a set E of elements, not necessarily numbers; (B) the set of functional values of f is convex; (C) a quasi-functional relation exists among $v_1, v_2, \cdots, v_n$ where

$$v_r=v_r(x_1, x_2, \cdots, x_n)=u_r[f(x_1), \cdots, f(x_n)]$$

and the set $u_r(t_1, t_2, \cdots, t_n)$ ($r=1, 2, \cdots, n$) is a normal set of functions over an open set O containing $[f(x_1), \cdots, f(x_n)]$, $x_r \in E$. *R. L. Jeffery.*

Foulkes, H. O. Theorems of Kakeya and Pólya on power-sums. Math. Z. 65 (1956), 345–352.

Consider an additive semigroup A of positive integers and $\nu_1, \nu_2, \cdots, \nu_n$ the first n positive integers not belonging to A. There is a theorem of S. Kakeya [Jap. J. Math. 2 (1925), 69–80; 4 (1927), 77–85] to the effect that any symmetric function of n indeterminates x_i ($i=1, 2, \cdots, n$) can be expressed as a rational function of a fundamental set $s_{\nu_1}, s_{\nu_2}, \cdots, s_{\nu_n}$, where $s_k=\sum_{i=1}^{n} x_i^k$. Kakeya's proof is not a constructive one. The author gives a short constructive proof of Kakeya's theorem and gives its application to a result of Pólya [J. Math. Pures Appl. (9) 31 (1952), 37–47; MR 13, 841]. *B. W. Jones.*

Knobloch, Hans-Wilhelm. Die Seltenheit der reduziblen Polynome. Jber. Deutsch. Math. Verein. 59 (1956), Abt. 1, 12–19.

Let

$$F(x, t)=F(x_1, \cdots, x_k; t_1, \cdots, t_s)$$

denote an irreducible polynomial in the $k+s$ indeterminates x_i, t_j with integral coefficients; let $R(N)$ denote the number of integral s-tuples $\tau_1, \cdots, \tau_s$ with $|\tau_i|\leq N$ such that $F(x_1, \cdots, x_k; \tau_1, \cdots, \tau_s)$ is reducible. In a previous paper [Abh. Math. Sem. Univ. Hamburg 19 (1955), 176–190; MR 16, 798] the writer showed that the ratio

$$(*) \qquad (2N+1)^{-s}R(N)=O(N^{-\alpha}),$$

where $\alpha>0$. This result is now improved in the following way. Let $F(x, t)$ be normalized and of degree n in one of the x_i and of degree $\leq l_j$ in t_j, where $l_j\leq l_s$ for $j<s$. Then α in (*) satisfies

$$\alpha\geq\frac{1}{9(n-1)}\left\{\sum_1^s l_j+2l_s\sum_1^{s-1} l_j\right\}^{-1}.$$

Finally if $R_1(N)$ denotes the number of normalized polynomials of fixed degree m, with coefficients between $-N$ and $+N$, and Galois group of order $<m!$, then it is proved that

$$\frac{R_1(N)}{(2N+1)^m}=O(N^{-\beta}), \quad \beta=\frac{1}{18m(m!)^3}.$$

L. Carlitz (Durham, N.C.).

Hoffman, A. J.; Newman, M.; Straus, E. G.; and Taussky, O. On the number of absolute points of a correlation. Pacific J. Math. 6 (1956), 83–96.

Let π be a finite projective plane of order n($n+1$) points on each line). Here $n\geq 2$ and the number of points (lines) of π is $N=n^2+n+1$. The authors assume that π possesses a correlation ρ (a one-to-one mapping of the N points upon lines and the N lines upon points which preserves incidence) and concern themselves with the number, M, of absolute points of ρ (that is, the number of instances of a point P lying on its image ρP). The problem has previously been considered by Ball [Duke Math. J. 15 (1948), 929–940; MR 10, 469] and, for the case of a polarity ($\rho^2=1$), by Bear [Bull. Amer. Math. Soc. 52 (1946), 77–93; MR 7, 387]. The present paper is new in two respects: (a) The problem is reduced to algebraic number theory through the use of the incidence matrix. (b) The methods carry over to the more general subject of symmetric group divisible designs.

The authors first treat the following problem. Let $f(x)$ be a polynomial with integral coefficients which is irreducible over the field of rationals and suppose that f has a zero of form $n^{1/k}\zeta$ where n, k are natural numbers and ζ is a root of unity. Then, for some natural number h, $f(x)$ is a divisor of $n^{\phi(h)}\Phi_h(x^k/n)$, where ϕ is Euler's totient and $\Phi_h(x)$ is the cyclotomic polynomial corresponding to the primitive hth roots of unity. Attention is restricted to the case $k=2$ (for $k>2$, see the paper by Straus and Taussky reviewed below). Putting $n=(n^*)^2n'$

where n^*, n' are nutural numbers and n' is square-free, the authors prove the following (Lemma 2): $\Phi_h(x^2/n)$ is reducible if and only if $n'|h$ and one of the following holds: (a) $h\equiv 1 \pmod 2$ and $n'\equiv 1 \pmod 4$; (b) $h\equiv 2 \pmod 4$ and $n'\equiv 3 \pmod 4$; (c) $h\equiv 4 \pmod 8$ and $n'\equiv 0 \pmod 2$. In each case of reducibility, $n^{\phi(h)}\Phi_h(x^2/n)=\pm g(x)g(-x)$, where $g(x)$ is an irreducible polynomial distinct from $g(-x)$. The authors go on to prove Lemma 3: The sum of the zeros of $g(x)$ is: (a) $\pm n^*n'$ if $h\not\equiv 0 \pmod 4$ and h is square-free; (b) 0 if $h\not\equiv 0 \pmod 4$ and h is not square-free; (c) $\pm n^*n'$ if $h\equiv 0 \pmod 4$ and $h/4$ is odd and square-free; (d) 0 if $h\equiv 0 \pmod 4$ and $h/4$ is odd and not square-free.

At this point the authors are ready to handle the problem discussed in the first paragraph. In terms of an arbitrary ordering $P_1, \cdots, P_N$ of the N points of π, they define an incidence matrix $A=(a_{ij})$ and a permutation matrix $P=(p_{ij})$ as follows: $a_{ij}=1$ or 0 according as P_i lies or does not lie on ρP_j; $p_{ij}=1$ or 0 according as P_i equals or does not equal $\rho^2 P_j$. Then: (i) $A^T=PA$; (ii) $M=\operatorname{tr} A$; (iii) $AA^T=A^TA=nI+J$, where I is the identity matrix and J has every element equal to 1. Since P is a permutation matrix, its characteristic roots are roots of unity completely given by the orders $d_1, \cdots, d_r$ of the cycles of P. One characteristic root of A is $n+1$. By (i), (iii), the others have form $n^{\frac{1}{2}}\zeta$, where ζ is a root of unity. This is enough to prove the known result that $M\equiv n+1 \pmod{n^*n'}$. To penetrate more deeply, the authors note $M=n+1-a$, where $(x-n-1)Q(x)$ is the characteristic polynomial of A and $Q(x)=x^{N-1}+ax^{N-2}+\cdots$. Moreover, each irreducible factor of $Q(x)$ is a factor of $n^{\phi(h)}\Phi_h(x^2/n)$, where h can range over $d_1, \cdots, d_r$. By ensuring (in terms of Lemma 2) that the latter polynomial is irreducible for each admissible h, they prove that $a=0$ and hence $M=n+1$; their results subsume those of Ball in the paper cited above. Finally, they allow some reducibility and obtain bounds for the integer s in the equation $M=n+1+sn^*n'$ by application of Lemma 3. A concluding section of the paper sketches the corresponding theory for the case of symmetric group divisible designs.

R. H. Bruck (Madison, Wis.).

Straus, E. G.; and Taussky, O. Remark on the preceding paper. Algebraic equations satisfied by roots of natural numbers. Pacific J. Math. **6** (1956), 97–98.

The authors consider the polynomial $\Phi_h(x^k/n)$, whose factorization for $k=2$ is determined in the paper reviewed above. They note first that if $n=m^d$, where $d|k$, and if y is defined by $my=x^{k/d}$, then $\Phi_h(x^k/n)=\Phi_h(y^d)$ and the latter is either reducible with cyclotomic factors or equal to $\Phi_{hd}(y)$. Hence they exclude this case. This understood they prove that $\Phi_h(x^k/n)$ is reducible precisely when k is even and $\Phi_h(x^2/n)$ is reducible; in which case, $\Phi_h(x^k/n)=g(x^t)g(-x^t)$, where $t=k/2$ and the polynomials $g(\pm x^t)$ are irreducible.

R. H. Bruck (Madison, Wis.).

See also: Derwidué, p. 212.

Partial Order Structures

Sedmak, Victor. Quelques applications des ensembles ordonnés. Bull. Soc. Math. Phys. Serbie **6** (1954), 12–39, 131–153.

This is a self-contained, leisurely, and detailed account of the author's research on the dimension, in the sense of Dushnik-Miller, of polyhedra. Most of the results have been announced previously [Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **7** (1952), 169–182; C. R. Acad. Sci. Paris **236** (1953), 2139–2140; MR **14**, 783; **15**, 50]. A large number of examples are given.

E. Hewitt (Seattle, Wash.).

Funayama, Nenosuke. Imbedding partly ordered sets into infinitely distributive complete lattices. Tôhoku Math. J. (2) **8** (1956), 54–62.

Some results of the reviewer and McLaughlin [Duke Math. J. **19** (1952), 683–693; MR **14**, 717] on infinite distributive lattices are generalized to partially ordered sets. In addition the author studies the class of imbedding operators which imbed a partially ordered set in an infinitely distributive complete lattice and determines the strongest such operator.

R. P. Dilworth (Pasadena, Calif.).

Dean, R. A. Completely free lattices generated by partially ordered sets. Trans. Amer. Math. Soc. **83** (1956), 238–249.

The author studies the lattices FL(P) (the free lattice generated by the partially ordered set P and preserving bounds of pairs of elements) and CF(P) (the free lattice generated by P and preserving only order) which were introduced by the reviewer in the study of lattices with unique complements [same Trans. **57** (1945), 123–154; MR **7**, 1]. It is shown that many of the results of Whitman [Ann. of Math. (2) **42** (1941), 325–330; **43** (1942), 104–115; MR **2**, 244; **3**, 261] can be extended to the completely free lattices CF(P). In particular, the word problem is solved for these lattices and the existence of canonical forms is proved. Although analogous results do not hold in general for F(P), there are important cases in which F(P)=CF(P) and the results do apply. Thus when P is an unordered sum of chains, F(P)=CF(P) and the word problem has a solution even though the methods of Evans cannot be applied directly.

R. P. Dilworth.

Dwinger, Ph. Direct products in modular lattices. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 435–443.

The author studies the direct sum and product decompositions in complete modular lattices in which the finite chain condition does not hold. Baer has given a condition under which the main theorem holds that if indecomposable direct sum representations exist the components are isomorphic in pairs [Trans. Amer. Math. Soc. **64** (1948), 519–551; MR **10**, 425]. This condition is not self-dual but it is shown that a direct sum decomposition also gives rise to a direct product representation for which the theorem holds. It is also shown that the condition of Baer implies a condition used by Kurosch [Theory of groups, 2nd ed., Gostehizdat, Moscow, 1953, Ch. 11; MR **15**, 501].

O. Ore (New Haven, Conn.).

Andreoli, G. Geometrie booleane affini: calcolo matriciale e vettoriale su Algebre di Boole. Giorn. Mat. Battaglini (5) **3(83)** (1955), 137–156 (1956).

Der Verfasser untersucht Boolesche Matrizes von n Zeilen und n Spalten, die aus den Elementen einer endlichen Booleschen Algebra gebildet sind. Unter einer Elementarmatrix versteht er eine solche, deren jedes Element entweder dem Nullelement N oder dem Universalelement T der Booleschen Algebra gleich ist. Das System aller Elementarmatrizes ist in bezug auf die Addition und Multiplikation abgeschlossen. Jede Matrix kann als eine lineare Kombination von Elementarmatrizes geschrieben

werden, deren Koeffizienten den Atomen der Booleschen Algebra gleich sind. Unter einer Substitutionsmatrix wird eine Elementarmatrix verstanden, in welcher $\nu \leqq n$ Elemente gleich T sind und welche die Eigenschaft hat, dass in keiner Zeile und keiner Spalte T mehr als einmal auftritt. Ist $\nu = n$, so wird diese Matrix eigentlich, sonst uneigentlich genannt. Eigentliche Substitutionsmatrizes bilden in bezug auf die Multiplikation eine Gruppe, uneigentliche ein Gruppoid. Eine Matrix wird als orthonormal bezeichnet, wenn das Produkt zweier verschiedenen Elemente derselben Zeile oder derselben Spalte gleich N, die Summe aller Elemente einer Zeile oder Spalte gleich T ist. Diese Matrizes bilden in bezug auf die Multiplikation eine Gruppe; das inverse Element zu einer solchen Matrix ist ihre Transponierte.

Das geordnete System von n Elementen der Booleschen Algebra heisst Vektor; ein Vektor wird als eine Matrix mit einer Spalte und n Zeilen aufgefasst. Ein Vektor heisst orthonormal, wenn das Produkt von zwei verschiedenen Elementen des Vektors gleich N und die Summe aller Elemente des Vektors gleich T ist. Das Produkt einer ortho normalenMatrix und eines orthonormalen Vektors ist ein orthonormaler Vektor. Es wird das Skalarprodukt zweier Vektoren definiert und studiert. Es werden die Vektoren- und Matrizenrechnungen verglichen, welche zu zwei Booleschen Algebren von verschiedener Anzahl von Elementen konstruiert sind. Die eingeführten Begriffe werden geometrisch interpretiert. Die Literaturangaben fehlen, obwohl einige Resultate bekannt sind [vergl. Wedderburn, Ann. of Math. (2) 35 (1934), 185–194]. Einige Druckfehler erschweren das Studium der Arbeit. *M. Novotný* (Brno).

Mori, Tuyosi. On the group structure of Boolean lattices. Proc. Japan. Acad. 32 (1956), 423–425.

I. Amemiya and the author have previously studied Boolean lattices, considered as groups under the operation of symmetric difference. Now, theorems of Naimark—B.Sz. Nagy on the extension of operator-valued functions, defined on a topological group, are shown to hold with a Boolean lattice in place of the group. Another result: an operator ring is isomorphic to a product of factors of type 1 and of finite class if and only if every commutative subgroup of its unitary group whose elements are of order 2 has sufficiently many continuous characters.

I. Halperin (Kingston, Ont.).

See also: Pekelis, p. 189; Birkhoff and Pierce, p. 191.

Rings, Fields, Algebras

Pollak, D. On Abelian algebras. Uč. Zap. Kazan. Univ. 115 (1955), no. 14, 145–156. (Russian)

Let G be the Galois group of a field extension Σ of Ω. It is well-known that for G cyclic, each non zero element of Ω determines an associative algebra (a crossed product of Σ and G). Two such elements, α, $\tilde{\alpha}$ determine isomorphic algebras when $\alpha\tilde{\alpha}^{-1}$ is a norm $N(z)$, $z \in \Sigma$. The author shows that when G is a product of cyclic groups G_i $(1 \leqq i \leqq m)$, each crossed product is determined by $\frac{1}{2}m(m+1)$ elements α_i $(1 \leqq i \leqq m)$ and z_{ji} $(2 \leqq j \leqq m)$, $(1 \leqq i \leqq j-1)$ of Σ, satisfying the following conditions:

$$\alpha_i^{S_i} = \alpha_i, \quad \alpha_j^{S_i}/\alpha_j = N_j(z_{ji}), \quad \alpha_i/\alpha_i^{S_j} = N_i(z_{ji}),$$
$$z_{ki}^{S_i}/z_{ki} = (z_{kj}^{S_j}/z_{kj})(z_{ji}^{S_k}/z_{ji})$$

(S_i a generator of G_i and $k > j > i$). Two sets of such elements determine isomorphic algebras if

$$\tilde{z}_{ji} = (c_j^{S_j} c_i / c_j c_i^{S_j}), \quad \tilde{\alpha}_i = \alpha_i N_i(c_i) \quad (c_i \in \Sigma).$$

{Reviewer's remark: This results can be obtained out of the Künneth relations applied to the product of the canonical complexes for cyclic groups described by N. E. Steenrod [Proc. Nat. Acad. Sci. U.S.A. 39 (1953), 217–223; MR 14, 1006].} *E. Lluis* (Mexico, D.F.).

Ribenboim, P. Sur la théorie des idéaux dans certains anneaux de type infini. An. Acad. Brasil. Ci. 28 (1956), 21–39.

Let R be a commutative Noetherian ring with identity. By R_Λ we understand the ring of polynomials in X_a $(a \in \Lambda)$ with coefficients in R, where Λ is a set of indices and the X_a are independent indeterminates over R. Let λ be a finite subset of Λ and let R_λ be the ring of those elements of R_Λ in which only X_a with $a \in \lambda$ are involved. We set $A_\lambda = A \cap R_\lambda$ when A is an ideal in R_Λ. In the set I of all ideals of R_Λ a neighbourhood system is defined as follows. When A is in I then every λ defines a neighbourhood of A consisting of all B in I with $B_\lambda = A_\lambda$. It is shown: (i) Those ideals of R_Λ which are the intersection of a finite number of primary ideals form a subset of I which is everywhere dense in I. (ii) R_Λ is a Jacobson-ring [as defined by Krull in Math. Z. 54 (1951), 354–387; MR 13, 903] when R is a field and when the maximal ideals in R_Λ form a closed set in I. (iii) With certain restriction on R the set of those v-ideals [see Prüfer, J. Reine Angew. Math. 168 (1932), 1–36] in R_Λ which are an intersection of a finite number of primary v-ideals is everywhere dense in the set of all v-ideals of R_Λ. *F. J. Terpstra.*

Szász, F. On rings every subring of which is a multiple of the ring. Publ. Math. Debrecen 4 (1956), 237–238.

Rings R for which the additive groups are cyclic are characterized as those for which every subring has the form nR, where n is an integer. [See Szász, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 5 (1955), 491–492; Acta Math. Acad. Sci. Hungar. 6 (1955), 475–477; MR 17, 709, 940]. *F. Haimo* (St. Louis, Mo.).

Curtis, C. W. On commuting rings of endomorphisms. Canad. J. Math. 8 (1956), 271–292.

Let M, M' be right and left modules, respectively, over a ring B, and let there be a B-bilinear and non-degenerate function τ: $M' \times M \to B$ such that the submodule $b = \tau(M', M)$ of B has a two-sided identity element. Set $x(\psi \odot u) = u\tau(\psi, x)$ for x, $u \in M$, $\psi \in M'$, and let C be any subring of B-endomorphism ring E of M containing $M' \odot M$. The paper proves, among other things, that the C-endomorphism ring of M is the set of B-operations and that $R \to \tau(M', R)$ gives 1-1 map, preserving direct sums, intersections and isomorphism relation, between direct C-summands R of M and left ideal direct components of the ring b. If B satisfies the minimum condition, each R contains a unique maximal C-submodule S and a C-isomorphism $R_1/S_1 \cong R_2/S_2$ implies a C-isomorphism $R_1 \cong R_2$. A similar result holds when we replace C, b, τ with B, $M' \odot M$, $\odot$. The main application of the theory is to projective representations of a finite group by semilinear transformations. Also Weyl's [Duke Math. J. 3 (1937), 200–212] result is seen to be its consequence.

T. Nakayama (Nagoya).

Fuchs, L. Ringe und ihre additive Gruppe. Publ. Math. Debrecen 4 (1956), 488–508.

In the literature there are already some results concerning the structure of the additive group of a ring [see Beaumont; MR 10, 10; Goldman, MR 8, 433; Rédei, MR 17, 453; Rédei and Szele, MR 12, 155; Szele, MR 11, 496; 13, 316; 17, 122], so far, however, a systematic investigation of the additive group of rings was wanting. The author pursues his investigations in two directions: to a given class A of abelian groups he seeks to determine the corresponding class B of rings, namely the class of rings the additive group of which belongs to A; conversely, given a class B of rings, the corresponding class A of groups is to be determined, namely the class of groups on which rings belonging to B can be built. The main result concerning the first problem essentially answers the question for abelian torsion groups. It is shown that if the additive group R^+ of the ring R is a p-group, then the multiplication of the elements of R is already determined by the multiplication of elements of any basic subgroup of R^+. Starting with this, it is shown that the rings which can be built on a given torsion group are capable of a comparatively explicit characterization. As to the second problem the author gives a complete description of those classes of abelian groups which belong to the following classes of rings: simple rings, rings with descending chain condition for left-ideal, semi-simple rings, i.e. rings containing no nonzero nilpotent left-ideal and satisfying the descending chain condition for left-ideals, torsion rings with ascending chain condition for left-ideals, torsion rings without annullators, torsion rings with a unit element. (A ring is called a torsion ring if its additive group is a torsion group.) Other classes of rings are also investigated. *A. Kertész* (Debrecen).

Bourne, Samuel. On multiplicative idempotents of a potent semiring. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 632–638.

A semiring is a system S that is a semigroup under addition, a semigroup under multiplication, and in which the usual distributive laws hold. In this article, the author considers only semirings S with an identity element 0 for addition, and after giving examples to show that it involves some loss of generality, makes the assumption throughout the remainder of the paper that $0S=S0=0$ for all $s\in S$. A right ideal R of S is a subset of S that contains 0, is closed under addition, and such that $r\in R$, $s\in S$ implies $rs\in R$. Left and two-sided ideals are defined similarly, and the terms multiplicative idempotent and nilpotent ideal are defined in the natural way. A semiring is potent if it contains no nilpotent left or right ideals.

A number of theorems about potent semirings in which every two-sided ideal contains a (nonzero) minimal left ideal and a minimal right ideal are proved. His main conclusion about such semirings is that every nonzero right ideal contains a (nonzero) multiplicative idempotent. {Reviewer's remark. A little care should be taken in regarding the author's theorems as generalizations of the corresponding (known) theorems about rings. For, a ring without a unit, when considered as a semiring, may have ideals in the author's sense that are not ideals in the usual ring-theoretic sense. On the other hand, the author's proofs can probably be easily modified to take care of this matter.}

M. Henriksen (Princeton, N.J.).

Kovács, László. A note on regular rings. Publ. Math. Debrecen 4 (1956), 465–468.

A ring R is called regular in the sense of von Neumann [Proc. Nat. Acad. Sci. U.S.A. 22 (1936), 707–713] if for every element $a\in R$ there exists an element $x\in R$ such that $axa=a$. The following theorems are proved: 1. An arbitrary ring R is regular if and only if $JL=J\cap L$ holds for every right-ideal J and left-ideal L of R. 2. For a ring R the following conditions are equivalent: a) R is a regular ring without nonzero nilpotent elements; b) every quasi-ideal of R is idempotent. 3. A ring R is a (discrete) direct sum of division rings if and only if R satisfies the descending chain condition for principal (two-sided) ideals and every quasi-ideal of R is idempotent. A submodule M of the ring R is called a quasi-ideal if $RM\cap MR\subseteq M$ [Steinfeld, Acta Math. Acad. Sci. Hungar. 4 (1953), 289–298; MR 16, 992]. *A. Kertész.*

Kostrikin, A. I. On Lie rings satisfying the Engel condition. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 580–582. (Russian)

A Lie ring $\mathfrak{L}$ is said to satisfy the condition $E_{n,p}$ provided that $\mathfrak{L}$ has characteristic $p\geqq 0$, that $\mathfrak{L}$ satisfies the nth Engel condition (symbolically,

$$[\cdots[[u, v], v], \cdots, v]=0$$

for all u, $v\in\mathfrak{L}$), and that $n<p$ (unrestricted n in case $p=0$). Higgins [Proc. Cambridge Philos. Soc. 50 (1954), 8–15; MR 15, 596] discussed the nilpotency of Lie rings satisfying $E_{n,p}$ for certain values of n and p. The author shows that for $n=3$, 4, 5, a Lie ring satisfying $E_{n,p}$ is locally nilpotent. *R. A. Good* (College Park, Md.).

Borevič, Z. I.; and Faddeev, D. K. Theory of homology in groups. I. Vestnik Leningrad. Univ. 11 (1956), no. 7, 3–39. (Russian)

This paper is expository and treats the following subjects (along with full proofs): Classical definitions of cohomology (homology) of groups. Resolutions, independence of the cohomology (homology) groups of the specific resolution used. Cohomology sequence for finite groups, relations between the cohomology of a finite group and its Sylow subgroups. Reduction theorems [see, e.g., Borevič, Dokl. Akad. Nauk SSSR (N.S.) 104 (1955), 5–8; MR 17, 583]. *W. T. van Est* (Utrecht).

See also: van Albada, p. 184; Dean, p. 186; Birkhoff and Pierce, p. 191; Linnik, p. 194; Mostowski, p. 196; Šilov, p. 201; Samuel, p. 221; Słowikowski and Zawadowski, p. 223; Nagata, p. 224; Lenz, p. 228; Busemann, p. 230; Teleman, p. 232; Archbold and Johnson, p. 244; Iséki and Miyanaga, p. 223; Iséki, p. 223.

Groups, Generalized Groups

★**Kurosh, A. G. The theory of groups. Translated from the Russian and edited by K. A. Hirsch. Volume II.** Chelsea Publishing Company, New York, N.Y., 1956. 308 pp. $4.95.

Translation of Parts III and IV (chapters 9–15) of Teoriya Grupp, 2nd edition [see MR, 15, 501]. In the appendices (15 page) the translator has carried the subject matter of the text further to the present state of knowledge by surveying the relevant research of the last few years.

The new bibliography (21 pages) lists the group-theoretical papers to the end of 1955 and in some instances even beyond that date.

Pekelis, A. S. Lattice isomorphism of groups possessing a finite rational series. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 4 (70), 143–147. (Russian)

Let ϕ be a lattice isomorphism between the torsion-free groups G and G^ϕ. The mapping ϕ preserves each of the following properties: (1) a subgroup of G is an isolated abelian normal subgroup, (2) a subgroup of G is an isolated normal subgroup and has an ascending isolated series which is invariant in G and has abelian factors, (3) the group G itself possesses a finite rational series. Indeed, in the latter two cases, the images in G^ϕ of the series in G are series with the respective properties. *R. A. Good.*

★**Specht, Wilhelm. Gruppentheorie.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. vii+457 pp. DM 66.00.

This book was at first so impressive in its wealth of material that the reviewer regarded it as a complete encyclopedia on the subject. Soberer thoughts showed otherwise; it stands on a par with the well-known texts by Burnside, Speiser and Zassenhaus and the more recent volumes of Kuroš, but it does not supersede any of them. Rather than compare and contrast these books in detail, we shall quote some timely words from the Foreword: "An austere selection of material is, for all that, necessarily a function of personal taste; this truth must console the connoisseur if he misses this or that which lies in his heart." The reviewer grieves most for a lovely bower of theorems centred around representations by linear transformations and associated with the names of Schur and Burnside and Frobenius and Jordan and E. Noether; and he misses the theory of the transfer; but he is comforted by much that is new.

The book is divided into three parts of roughly 150 pages each, with chapter headings as follows: Part One: Introduction. 1.1. Foundations. 1.2. Subgroups of a group. 1.3. Homomorphy and isomorphy. 1.4. Groups with operators. Part Two: Free and direct decomposition. 2.1. The free group. 2.2. Free decomposition. 2.3. Direct decomposition. 2.4. Theory of abelian groups. Part Three: General structure theory. 3.1. Theory of normal series. 3.2. Theory of $\mathfrak{p}$-groups. 3.3. Extension theory. The bibliography takes the form of a commentary, entitled "Remarks and hints", which deals with each section separately. The index of Names is limited to 41 names which the author has attached in parentheses to theorems in the main body of the text. The Index of Terms contains about 400 items and seems to be complete.

Chapter 1.1 begins with a rapid survey of most of the concepts associated with sets, including Bernstein's Theorem, Zorn's Lemma, index sets, well-ordering, but with no explicit mention of transfinite cardinals or ordinals. Next come (single-valued) mappings, followed by semigroups and groups. The chapter ends with a rich offering of "examples", the most noteworthy of these being the definition of a free group of arbitrary rank in terms of mappings. The variety of Chapter 1.2 will best be indicated by selected topics: Dedekind's Theorem. Lagrange's Theorem. The symmetric and alternating groups of "finite" permutations of an arbitrary set. Transitive, multiply transitive and imprimitive groups. Dietzman's Theorem (a finite, order-finite normal complex generates a finite normal divisor). Inner automorphisms. Conjugate classes—complete details for symmetric groups. Simplicity of the alternating group on a set with more than four elements. Simplicity of the union of an inductively ordered set of simple subgroups of a group. Cauchy's Theorem. A finite p-group has a non-trivial centre. A non-abelian group has a non-cyclic central quotient group. Determination of all groups of orders p^2 and p^3. — It may be taken for granted that the rest of the book is equally varied; we shall select a few topics for discussion.

The last three sections of Chapter 1.4 contain a novel and systematic treatment of group properties which we sketch here in simplified form. An e-group is a group with property e. We deal only with properties which are abstract; that is, preserved under isomorphism. Each property e determines a property le: an le-group, or local e-group, is a group all of whose finitely generated subgroups are e-groups. The author considers the following axioms: (I) The group with one element is an e-group. (II) Each subgroup of an e-group is an e-group. (III) Each homomorphic image of an e-group is an e-group. (IV) If G is a group with a normal subgroup N such that N and G/N are e-groups, then G is an e-group. (V) The union of any chain of e-subgroups of a group is an e-group. (VI) The subgroup generated by any two e-subgroups of a group is an e-group. In addition, (II) and (V) are weakened to (II*) and (V*) by replacing "subgroup" by "normal subgroup"; similarly, (VI) becomes (VI*) or (VI**) according as one or both instances of "e-subgroup" are replaced by "normal e-subgroup". There are many theorems about properties satisfying some subset of the axioms; significant examples attest to the economy of the method. If e has type (I, II), e-groups are le-groups. According as e has type (I, II) or (I, III) or (I, II, VI), le has type (I, II, V) or (I, III) or (I, II, V, VI). To give some examples, the property of satisfying the maximal condition has type (I, II, III, IV, VI*); that of being locally finite satisfies all the axioms. Other significant properties are those of being finite, order-finite, class-finite, torsion-free, free or locally free or abelian. Locally finite groups are order-finite; the question as to the converse is the generalized Burnside problem proposed by Kuroš. A property s of type (I, II, III, IV, V) is called a Sylow property; maximal s-subgroups exist in every group, are called s-Sylow-subgroups and have properties worthy of their name. For example, an s-Sylow-subgroup is unique and fully invariant in its normalizer. A property e of type (I, II*, III, IV) permits a "duality principle" in the following sense: If G is a group, the set of all normal e-subgroups of G is a lattice closed under unrestricted intersection; and the set of all normal subgroups N of G such that G/N is an e-group, is a lattice closed under unrestricted union. Examples show that the "principle" cannot be strengthened without further hypotheses. — The author's theory is actually in terms of properties defined on groups with a preassigned set of operators and hence is much richer than the above sketch might indicate.

Now we turn to Part Two. The most noteworthy feature of Chapters 2.1, 2.2 seems to be a meticulous study of word subgroups and, in particular, of the higher commutator subgroups. In Chapter 2.3, the following simple lemma (Satz 7) seems to be new: If θ, ϕ are normal, idempotent operator-endomorphisms of an operator group G, the commutator-mapping $(\theta, \phi)=\theta\phi-\phi\theta$ is an operator-endomorphism of G into the centre of G. The chapter constitutes a remarkably up-to-date treatment of

direct decomposition into finitely or infinitely many factors, and the lemma frequently intervenes to simplify the known proofs. Chapter 2.4 contains, in thirty-seven pages, a large part of the lore of abelian groups, but the author himself refers the reader to the more extensive account by Kaplansky [Infinite abelian groups, Univ. of Michigan Press, 1954; MR **16**, 444].

The formal theory of abstract group properties, which played a minor rôle in Part Two, comes into its own in Part Three. Chapter 3.1 begins with the Schreier refinement theorem (and the appropriate form of the Jordan-Hölder theorem) for various types of (transfinite) ascending normal series. If e is an abstract group property, a group G is called an e_ω-group provided G has a (transfinite) ascending chief series (each term normal in G) whose quotient groups are all e-groups. If e has type (I, II, III), then e_ω has type (I, II, III, IV, V*, VI*). In this case there are, among others, the following consequences: (1) Every group G contains a unique subgroup $V=V(G)$ which is maximal in the property of being a normal e_ω-subgroup of G. (2) G is an e_ω-group if and only if every homomorphic image $H\neq 1$ of G has a normal e-subgroup $N\neq 1$. (Similarly, e defines a property in terms of (transfinite) descending chief series; the theory is not as satisfactory as that for the ascending case.) The author next examines various special cases of e, considering in greatest detail the metabelian and metacyclic groups. Solvability is defined in terms of the (transfinite) derived series (of iterated commutator subgroups).

So far we have been able to ignore operators; but operators are essential for the author's generalization of the concept of nilpotency. Let Ω be a given set of operators. An operator group F; Ω is called a fixed-group (relative to Ω) if Ω induces the identity transformation on F. An operator group G; Ω is called Ω-nilpotent (lower Ω-nilpotent) if G possesses an ascending (descending) normal series (of Ω-subgroups) whose quotient groups are all fixed-groups. If Ω happens to coincide on G with the group of inner automorphisms of G, we get the usual (upper and lower) concepts of nilpotency.

Chapter 3.1 concludes with a section on the theory of q-groups and u-groups: a group G is defined to be a q-group (u-group) if no proper quotient-group (subgroup) is isomorphic to G.

The $\mathfrak{p}$-groups of Chapter 3.2 are defined as follows: $\mathfrak{p}$ is an arbitrarily prescribed set of prime integers and a group is a $\mathfrak{p}$-group provided every element is a $\mathfrak{p}$-element; that is, an element whose order is finite and divisible only by primes in $\mathfrak{p}$. If $\mathfrak{p}$ consists of a single prime p, we have the familiar notion of a p-group. The chapter is mainly devoted to (not necessarily finite) p-groups, particularly in relation to conditions ensuring nilpotency, local nilpotency or local finiteness. The notion of a Sylow system of a group is also explored. A set Σ of subgroups of a group G is called a Sylow system of G provided Σ consists of $\mathfrak{p}$-Sylow-subgroups $S(\mathfrak{p})$, one for each prime-set $\mathfrak{p}$, such that $S(\mathfrak{p}\frown\mathfrak{q})=S(\mathfrak{p})\frown S(\mathfrak{q})$ and $S(\mathfrak{p}\smile\mathfrak{q})=S(\mathfrak{p})S(\mathfrak{q})$ for every pair of prime-sets $\mathfrak{p}$, $\mathfrak{q}$. A sufficient condition that a finite group G possess a Sylow system is that G be solvable; the condition is also necessary but the author, as a result of his decision to omit the theory of representations by linear transformations, finds himself unable to supply a proof. Among other theorems we may quote the following: (1) Every order-finite, class-finite metacyclic group possesses a Sylow system. (2) If an order-finite group G possesses a Sylow system with only finitely many conjugates, then every two Sylow systems of G are conjugate.

In Chapter 3.3, a group G is called an extension or normal extension of a group U if U is a subgroup or normal subgroup, respectively, of G. There is a detailed theory centering on various methods of classifying extensions .In contrast to certain recent trends, cohomology methods play no rôle in the theory.

We must emphasize, in conclusion, that many interesting and significant topics have gone without mention in this brief review. And, finally, we must comment somewhat unfavourably on the "Remarks and hints". Naturally enough, if we consider the passing of time alone, these bibliographical notes are much superior to anything comparable in Burnside or Speiser or Zassenhaus. On the other hand, they cannot stand comparison with the English translation of Kuroš "The theory of groups" [Chelsea, New York, 1955, 1956; MR **17**, 124; see also the preceding review]. For the latter has, in its two volumes, an appendix of more than 20 pages and a bibliography of roughly 50 pages to serve the same general purpose as Specht's 11 pages of "Remarks and hints".

R. H. Bruck (Madison, Wis.).

Łoś, J.; Sąsiada, E.; and Słomiński, Z. On abelian groups with hereditarily generating systems. Publ. Math. Debrecen 4 (1956), 351–356.

An abelian group is said to have a hereditarily generating system (h.g.s.) if it has an infinite system of generators such that every subsystem thereof which has the same power as a set is also a generating system for the group. A group with an h.g.s turns out to be countable. Any group of type p^∞ is an example. A torsion abelian group has an h.g.s. if and only it is the direct sum of an infinite number of cyclic groups of prime power orders with no repetition of primes and of (possibly) a countable divisible group. An aperiodic abelian group with an h.g.s. is of finite rank; and, conversely, a divisible aperiodic group of finite rank has an h.g.s. The non-existence of noncyclic homomorphic images suffices for the existence of an h.g.s. for a countable torsion group. It is an open question whether this same condition suffices in the case of torsionfree groups of finite rank. *F. Haimo.*

Honda, Kin'ya. Realism in the theory of abelian groups. I. Comment. Math. Univ. St. Paul. 5 (1956), 37–75.

A largely expository account of recent results on infinite abelian groups, enlivened by unusual terminology that begins with the title. *I. Kaplansky.*

Balcerzyk, Stanisław. Remark on a paper of S. Gacsályi. Publ. Math. Debrecen 4 (1956), 357–358.

A new proof of a theorem of Gacsályi [same Publ. 4 (1955), 89–92; MR **16**, 898]: if A is a subgroup of an abelian group G such that every system of linear equations over A solvable in G is also solvable in A, then A is a direct summand of G. *I. Kaplansky.*

Hall, P. Finite-by-nilpotent groups. Proc. Cambridge Philos. Soc. 52 (1956), 611–616.

Denote by ${}^iG$ and $Z_i(G)$ for $i=0, 1, 2, \cdots$ the terms of the descending and ascending central series of the group G. The main result of this investigation is the interesting theorem that one of the ${}^iG$ is finite if, and only if, one of the $G/Z_j(G)$ is finite. If $a=a(G)$ and $b=b(G)$ are minimal with the property that ${}^aG$ and $G/Z_b(G)$ are finite, then $a\leq b\leq 2a$; and there exist groups [of finite class] realizing a given pair of values a, b subject to the inequality $a\leq b\leq 2a$, though $a=b$ holds in many interesting cases.

If M is a normal subgroup of G, ${}_0M=M$, ${}_{i+1}M=[G, {}_iM]$, then more generally the finiteness of G/M and ${}_kM$ implies the finiteness of ${}^kG$ and $Z_{2k}(G)$. *R. Baer.*

Shephard, G. C. Some problems on finite reflection groups. Enseignement Math. (2) 2 (1956), 42–48.

After neatly summarizing the relevant work of H. Hopf [Ann. of Math. (2) 42 (1941), 22–52; MR 3, 61], Stiefel [Comment. Math. Helv. 14 (1942), 350–380; MR 4, 134], Coxeter [Duke Math. J. 18 (1951), 765–782; MR 13, 528], Borel [Ann. of Math. (2) 57 (1953), 115–207; MR 14, 490], Bott [Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 586–588; MR 16, 12], Shephard and Todd [Canad. J. Math. 6 (1954), 274–304; MR 15, 600], and Chevalley [Amer. J. Math. 77 (1955), 778–782; MR 17, 345], the author gives a simple proof that, for a finite group generated by unitary reflections of period 2, the Jacobian of the basic invariant forms factorizes into linear forms which, when equated to zero, give the reflecting hyperplanes. For a reflection of period p, the corresponding linear form occurs to the $(p-1)$th power. *H. S. M. Coxeter* (Toronto, Ont.).

Itô, Noboru. Über das Produkt von zwei zyklischen 2-Gruppen. Publ. Math. Debrecen 4 (1956), 517–520.

The author shows that if G is the product of two disjoint cyclic groups, A of order 2^a and B of order 2^b, then (1) if $c<a$, b, there is a corresponding normal subgroup N_c of G with factorization A_cB_c, each factor of order 2^c, where $A_c\subset A$ and $B_c\subset B$; (2) the derivative of G is not of type $(2^d, 2^d)$ although the author has previously proved it to be abelian [Math. Z. 62 (1955), 400–401; MR 17, 125]; (3) a cyclic center for G implies a cyclic derivative. [Reference is made to B. Huppert, ibid. 58 (1953), 243–264; MR 14, 1059.] *F. Haimo.*

Nakayama, Tadasi. A theorem on modules of trivial cohomology over a finite group. Proc. Japan Acad. 32 (1956), 373–376.

Let G be a finite group. A G-module A is said to be of trivial cohomology if the cohomology groups $H^n(K, A)$ vanish for all n (positive, negative, or 0) and all subgroups K of G. It is known, and important for applications in class field theory, that A is of trivial cohomology whenever there is an integer r such that $H^r(K, A)=(0)=H^{r+1}(K, A)$, for all subgroups K of G. The main result of the present paper is that, when G is a p-group, the conditions $H^r(G, A)=(0)=H^{r+1}(G, A)$ already imply that A is of trivial cohomology. This gives a considerable strengthening of the general result; in order to conclude that A is of trivial cohomology, one merely has to suppose that, for each prime p dividing the order of G, there is an integer $r(p)$, and a p-Sylow subgroup G_p of G, such that $H^{r(p)}(G_p, A)=(0)=H^{r(p)+1}(G^k, A)$.

A sketch of the proof is given, yielding the additional result that, if M is a G-module is which free and of finite rank as an abelian group, and if A is of trivial cohomology, so is the tensor product $A\otimes M$. Moreover, it is stated that this result holds, more generally, whenever M is a G-module without g-torsion, where g is the order of G.

Finally, it is shown that, when G is not a p-group, the main result, as stated first above for p-groups, can fail, and that it can also happen that $H^n(G, A)=(0)$, for all n, without A being of trivial cohomology.

G. P. Hochschild (Princeton, N.J.).

Birkhoff, Garrett; and Pierce, R. S. Lattice-ordered rings. An. Acad. Brasil. Ci. 28 (1956), 41–69.

The authors and this reviewer assume that the reader is familiar with the theory and terminology of lattice-ordered groups and vector lattices [see G. Birkoff, Ann. of Math. (2) 43 (1942), 298–331; Lattice theory, Amer. Math. Soc. Colloq. Publ. v. 25, rev. ed., New York, 1948; MR 4, 3; 10, 673]. An l-ring is a ring that is a lattice-ordered group and such that $a\geqq 0$ and $b\geqq 0$ imply $ab\geqq 0$, and an l-algebra is an l-ring which, regarded as a group, is a vector lattice over an ordered field K. The class of l-rings is equationally definable and so is the class of l-algebras. The l-radical N of an l-ring R is the set of all $a\in R$ such that for some positive integer n, $x_0|a|x_1|a|x_2\cdots x_{n-1}|a|x_n=0$ for all $x_0, x_1, \cdots, x_n\in R$. N is an l-ideal ($a, b\in N$, $c, d\in R$ and $|s|\leqq|a|$ imply $a\pm b$, $ac, ca, d\in N$) which is the union of the nilpotent l-ideals of R. If R is commutative or if R satisfies either chain condition for l-ideals, then the l-radical of R/N is 0. These results can easily be extended to l-algebras. No attempt is made to determine when the l-radical equals the ring radical.

The two dimensional real l-algebras are enumerated. In particular, the rational complex numbers do not admit an l-ordering. a commutative l-algebra with zero l-radical need not be the direct union of simple l-algebras, and a simple l-algebra need not be a full matrix algebra.

Let $E(R)$ be the ring of endomorphisms of the additive group of an l-ring R and $R^+=\{r\in R|r\geqq 0\}$. Then $E(R)$ is a partially ordered ring with positive elements

$$\{\alpha\in E(R)|R^+\alpha\subseteq R^+\}.$$

If $E(R)$ is an l-ring and the regular representation of R in $E(R)$ is a lattice isomorphism, then R is called regular. Group algebras and full matrix algebras are regular. Any homomorphic image of a real, finite dimensional, regular, Archimedean l-algebra is regular. An f-ring is an l-ring in which $a\frown b=0$ and $c\geqq 0$ imply $ca\frown b=ac\frown b=0$. Any f-ring is regular. An l-ring is an f-ring if and only if it is the subdirect union of simply ordered rings. An f-ring without zero divisors is simply ordered. Any Archimedean f-ring is commutative.

An l-ring R with a positive ring-unity 1 satisfies $a\frown b=0$ implies $ab=0$ if and only if 1 is a weak order unit. As one of the corollaries; an Archimedean l-ring with positive ring unity 1 is an f-ring if and only if 1 is a weak order unit. In any f-ring R the set Z_n of all elements a satisfying $a^n=0$ is an l-ideal. The l-radical of R is the join of these Z_n. An f-ring R, whose l-radical is zero and which satisfies the descending chain condition on l-ideals, is isomorphic to a direct union of l-simple ordered rings.

The last section deals with structure theorems for complete and σ-complete l-rings. For example, let A be a complete (σ-complete) l-algebra (not necessarily commutative or associative) with ring-unity 1 which is a strong order unit. Then A is isomorphic with the f-algebra $C(X)$ of all continuous functions on the Boolean space X associated with the complete (σ-complete) Boolean algebra of all components of 1.

There are numerous (at least 30) misprints in the paper and a few trivial errors in the statements of theorems and definitions. See for example, equations (1) and (4). The paper concludes with an interesting list of unsolved problems. *P. Conrad* (Colombo).

Klingen, Helmut. Über die Erzeugenden gewisser Modulgruppen. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. 1956, 173–185.

Let K be an algebraic field of degree s. Let $U_K{}^n$ be the group of n rowed matrices over K with determinant

which is a unit in K. It was proved by A. Hurwitz [same Nachr. **1895**, 332–356] that U_{K^n} has a finite set of generators. The author proves a similar result for the symplectic modular group over K. By symplectic modular group $M_n(K)$ we mean the group formed by $2n$ rowed matrices $\mathfrak{M}$ with integer elements of K and satisfying $\mathfrak{M}'\mathfrak{F}\mathfrak{M}=\mathfrak{F}$, where $\mathfrak{F}=\begin{pmatrix} 0 & I^{(n)} \\ -I^{(n)} & 0 \end{pmatrix}$. Similar result for rational field was proved previously by Hua and Reiner [Trans. Amer. Math. Soc. **65** (1949), 415–426; MR **10**, 684].

Let Σ be an imaginary quadratic field. He proves that the group formed by $2n$ rowed matrices $\mathfrak{M}$ with integer elements over Σ and satisfying $\overline{\mathfrak{M}}'\mathfrak{F}\mathfrak{M}=\mathfrak{F}$ has also a finite set of generators. *L. K. Hua* (Peking).

Kneser, Martin. Orthogonale Gruppen über algebraischen Zahlkörpern. J. Reine Angew. Math. **196** (1956), 213–220.

Improving considerably on previous results of the reviewer [Proc. London Math. Soc. (3) **2** (1952), 245–256; MR **13**, 820], the author almost brings to a close the question of the structure of the orthogonal and unitary groups over algebraic number fields K. He namely proves the following three theorems (and indicates without proof how corresponding results may be obtained for unitary groups): (A) [Notations are those of the reviewer's book "La géométrie des groupes classiques" [Springer, Berlin, 1955; MR **17**, 236].] For $n\geqq 3$, the factor group $O_n^+(K,f)/O_n'(K,f)$ is isomorphic to $K^*(f)/K^{*2}$, where $K^*(f)$ is the group of elements $\neq 0$ of K which are positive for all places at infinity $\mathfrak{p}_k$, where f is a definite quadratic form. (B) For $n\neq 4$, $O_n'(K,f)=\Omega_n(K,f)$. (C) For $n\geqq 5$, the group $\Omega_n/(\Omega_n Z_n)$ is simple. The only questions left open concern the orthogonal groups for $n=3$ and $n=4$, and the unitary groups for $n\leqq 4$. The methods of proof are of the same "geometrical" kind as those of the reviewer, but with many original and skillful details, and above all they use three new ideas, which make the progress possible. The first one consists in applying the Hasse theorems on representations of quadratic forms by other quadratic forms (whilst the reviewer only used the theorems on representation of numbers by quadratic forms), and using systematically the spinor norm. The other two are: 1) an approximation theorem for orthogonal transformations, similar to the classical theorem enabling one to approximate at once finitely many elements, each in a different local field $K_{\mathfrak{p}_k}$, by a single element of K; 2) the remark that in a local field $K_\mathfrak{p}$, two quadratic forms, whose corresponding coefficients differ by a sufficiently small quantity, are equivalent. For the details of the very concise proofs, we must refer to the paper itself.

J. Dieudonné (Evanston, Ill.).

Tamura, Takayuki; and Kimura, Naoki. Existence of greatest decomposition of a semigroup. Kōdai Math. Sem. Rep. **7** (1955), 83–84.

If E is a congruence relation over the semigroup S such that S/E is a semigroup of type T, then the authors call E a decomposition of S to a semigroup T. The relation $E\leqq F$ if $a\,Fb$ implies aEb for $a, b\in S$ is a partial ordering of the decompositions of S to a semigroup T. Let $T(\Lambda)$ be the variety of semigroups which satisfy a set of identities Λ. The authors show that there is, relative to the above partial ordering, a maximum decomposition of any semigroup S to a semigroup $T(\Lambda)$. *G. B. Preston.*

Griffiths, H. B. Infinite products of semi-groups and local connectivity. Proc. London Math. Soc. (3) **6** (1956), 455–480.

In this review semigroup means semigroup with identity. Then there is a natural retraction of the free product of two semigroups on either factor, and the unrestricted free product A_H of the sequence $\{A_i\}$ of semigroups can be defined, as it was for groups by the reviewer [J. London Math. Soc. **27** (1952), 73–81; MR **13**, 623]. If $\{B_i\}$ is a second sequence of semigroups, and γ_i is a homomorphism of B_i into A_i ($i=1, 2, \cdots$), there is a naturally defined homomorphism γ of B_H into A_H. Even if each γ_i is onto A_i, γ is not in general onto A_H. In particular, if each B_i is a free semigroup, γ maps B_H onto a subsemigroup A_T of A_H which the author calls the "topologists' product". If S_n denotes the subsemigroup of A_H generated by A_1, $A_2, \cdots, A_n$, and the product of the sequence $\{A_{n+i}\}$, then A_T can also be characterised as the intersection of all S_n. Lastly, if the A_i are all groups, let X_i be a metric space having A_i as fundamental group, and let the diameter of X_i tend to zero as i increases. Let the X_i have a single point in common, at which each is, in some suitable sense, locally connected. Then A_T is the fundamental group of their union. There are applications to the theory of local connectivity. *Graham Higman* (Oxford).

Hall, Marshall, Jr.; and Swift, J. D. Determination of Steiner triple systems of order 15. Math. Tables Aids Comput. **9** (1955), 146–152.

The authors state: "The procedures described in the paper represent, to the best of our knowledge, the first use of large scale computing equipment to isolate isomorphic algebraic systems." Without actually giving the codes (these are on file at Numerical Analysis Research University of California, Los Angeles) they describe the method by which SWAC was used to obtain the 80 non-isomorphic Steiner triple systems of order 15. They also suggest some improved procedures and briefly discuss the difficulties to be expected in the similar problem for order 19. It is a remarkable fact (pointed out by the authors) that, in 1915, F. N. Cole and his associates obtained the correct results for order 15 by hand calculation. {Reviewer's remark. The present results have an immediate interpretation for quasigroups; a Steiner triple system is essentially the same as an idempotent totally symmetric quasigroup in the sense of the reviewer [Trans. Amer. Math. Soc. **55** (1944), 19–52; **60** (1946), 245–354; MR **5**, 229; **8**, 134].} *R. H. Bruck* (Madison, Wis.).

Šik, František. Über Charakterisierung kommutativer Zerlegungen. Publ. Fac. Sci. Univ. Masaryk **1954**, 97–102. (Russian summary)

Let $\mathfrak{r}$, $\mathfrak{f}$ denote binary relations on a set G, or, when they are equivalence relations, $\mathfrak{r}$ and $\mathfrak{s}$ may equally well denote the corresponding decompositions of G into equivalence classes. Product $\mathfrak{r}\mathfrak{f}$ and supremum $\mathfrak{r}\vee\mathfrak{f}$ are defined as usual for relations. A set R of permutations on G relates any element a of G to each element into which a is carried by a permutation: this is called the relation belonging to R; if it is an equivalence relation, R (in this case a group) is called its basis.

Suppose that $\mathfrak{r}$, $\mathfrak{f}$ are equivalence relations with bases R, S. Then $\mathfrak{r}\mathfrak{f}=\mathfrak{f}\mathfrak{r}$ implies that $\mathfrak{r}\vee\mathfrak{f}=\mathfrak{r}\mathfrak{f}$ and that RS (the set of all $\alpha\beta$ where $\alpha\in R$, $\beta\in S$) is a basis of $\mathfrak{r}\mathfrak{f}$. The condition $\mathfrak{r}\mathfrak{f}=\mathfrak{f}\mathfrak{r}$ is shown to be equivalent to each of the following: (i) $\mathfrak{r}\mathfrak{f}$ is an equivalence relation, (ii) RS is a basis, (iii) the relation corresponding to RS is implied by

the supremum of the relations belonging to the cyclic groups generated by elements of $R\cup S$, (iv) the relations belonging to RS, SR are the same.

The theory is applied to quasigroups in the paper reviewed below. *I. M. H. Etherington.*

Šik, František. Über abgeschlossene Kongruenzen auf Quasigruppen. Publ. Fac. Sci. Univ. Masaryk **1954**, 103–112. (Russian summary)

This paper extends work of Kiokemeister [Amer. J. Math. **70** (1948), 99–106; MR **9**, 330], Thurston [Proc. London Math. Soc. (3) **2** (1952), 175–182; cf. also Proc. Amer. Math. Soc. **3** (1952), 10–12; MR **14**, 241; **13**, 621], and the author [Publ. Fac. Sci. Univ. Masaryk **1951**, 169–186; MR **15**, 7]. A congruence relation τ on a quasigroup G is by definition (1) right normal, (2) right subnormal, (3) right locally normal, (4) right hereditarily subnormal, (5) closed in G_r, the group generated by the translations of G, if the following conditions hold respectively: (1) $a, b, x \in G$, $ax\tau bx \Rightarrow a\tau b$, (2) $a, b \in G$, $ax\tau bx$ for all $x \in G \Rightarrow a\tau b$, (3) $\exists\, x \in G$ such that $a, b \in G$, $ax\tau bx \Rightarrow a\tau b$, (4) every finer congruence is right subnormal, (5) τ has a basis (cf. the preceding review) in G. Left concepts are defined similarly to (1) (2) (3) (4); normal means left and right normal, etc. The main results are as follows, with similar left theorems: The congruence τ is normal if and only if it has for basis a normal subgroup of G_r; τ is closed in G_r, and commutes with any normal congruence on G, if τ has property (3), or (4), or if G is a loop or a quasigroup whose centre contains all its right units.

I. M. H. Etherington (Edinburgh).

Kontorovič, P. G.; and Kacman, A. D. Some types of elements of a semigroup invariant in a group. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 145–150. (Russian)

Let G be a group with identity 1, containing no elements of finite order except 1. Let $\mathfrak{S}$ be a subsemigroup of G such that $1 \in \mathfrak{S}$ and such that $y^{-1}\mathfrak{S}y \subset \mathfrak{S}$ for all $y \in G$. Suppose also that $x \in \mathfrak{S}$ and $x \neq 1$ imply $x^{-1} \notin \mathfrak{S}$. A subset $\mathfrak{A}$ of $\mathfrak{S}$ such that $\mathfrak{S}\mathfrak{A} \subset \mathfrak{A}$ is an (left) ideal of $\mathfrak{S}$. For such $\mathfrak{A}$, it is easy to see that $\mathfrak{A}\mathfrak{S} \subset \mathfrak{A}$, and so all ideals of $\mathfrak{S}$ are 2-sided. An element $a \in \mathfrak{S}$ is said to be isolated if $x \in \mathfrak{S}$ and $x^n \in \mathfrak{S}a$ for some positive integer n imply that $x \in \mathfrak{S}a$. An element $a \in S$ is said to be indecomposable if $a = uv$ $(u, v \in \mathfrak{S})$ implies that $u=1$ or $v=1$. An element $a \in \mathfrak{S}$ is said to be prime if $(\mathfrak{S}a)' \cap \mathfrak{S}$ is a subsemigroup of $\mathfrak{S}$. A prime element is isolated and indecomposable, but examples can be found of elements that are isolated and indecomposable but not prime. Theorem 1. The set of all prime elements generates a commutative invariant subsemigroup of G. Theorem 2. An element $a \in \mathfrak{S}$ is indecomposable if and only if $\mathfrak{S} \cap \{a\}'$ is a subsemigroup of $\mathfrak{S}$. A basis for $\mathfrak{S}$ is a set of generators of $\mathfrak{S}$ each of which is indecomposable. Theorem 3. $\mathfrak{S}$ has a basis if and only if every proper subsemigroup of $\mathfrak{S}$ is contained in a maximal proper subsemigroup of $\mathfrak{S}$. Theorem 4. Let $\mathfrak{S}$ have a basis. Then the representation of an element in $\mathfrak{S}$ by elements of the basis is unique (except for order) if and only if every indecomposable element is prime. The authors give several other theorems, dealing with elements a, b for which $\mathfrak{S}a = \mathfrak{S}b$. The results of the present note are closely connected with those of two earlier papers of Kontorovič [Dokl. Akad. Nauk SSSR (N.S.) **93** (1953), 229–231; Kazan. Gos. Univ. Uč. Zap. **114** (1954), no. 8, 35–43; MR **15**, 681; **17**, 942].

E. Hewitt.

See also: Fuchs, p. 188; Gundlach, p. 195; Mahler, p. 196; Słowikowski and Zawadowski, p. 223; Edge, p. 227; Iséki and Miyanaga, p. 223; Iséki, p. 223.

THEORY OF NUMBERS

General Theory of Numbers

Mitsui, Takayoshi. On odd perfect numbers. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **6** (1956), 1–11.

The author calls an integer n a k-perfect number in case $\sigma(n) = 2kn$, where $\sigma(n)$ denotes the sum of the divisiors of n. It is proved that if $k \equiv 1 \pmod 4$ an odd k-perfect number n must have its decomposition into powers of primes of the form $p^\alpha q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_r^{2\beta_r}$, where p is of the form $8x+1$, $\alpha \equiv 4y+1$ and $q_i = 8z_i + 7$. For the case $k=1$ it is shown that $r > 2800$ and $\log n > 52729$. These results are obtained from an interesting use of Jacobi's formula for the number $R_4(n)$ of representations of an odd number n as the sum of four squares, namely

$$R_4(n) = 8\sigma(n).$$

Replacing the right member by $16kn$ and considering all possible cases of some of the four squares being equal, the author finds that

$$kn \equiv 2A + 3B \pmod 4,$$

where A and B are the numbers of sets of positive integers (x_1, x_2) satisfying $n = x_1^2 + 2x_2^2$ and $n = x_1^2 + x_2^2$ respectively. Since B is odd, $B=1$ and $A=1$ from which the results follow. Other results are obtained which show that certain extremely special odd numbers are not perfect.

D. H. Lehmer (Berkeley, Calif.).

Linnik, Yu. V. The asymptotic distribution of reduced binary quadratic forms in relation to the geometries of Lobačevskiĭ. I, II, III. Vestnik Leningrad. Univ. **10** (1955), no. 2, 3–23; no. 5, 3–32; no. 8, 15–27. (Russian)

Let Δ consist of those points (a, b, c) in three-dimensional Euclidean space such that $|2b| < a < c$ or $0 \leq 2b < a = c$ or $0 < 2b = a \leq c$. Let Σ be a convex cone with center at the origin lying entirely either in that portion of Δ for which $c \leq Ka$ or in that portion for which $c \geq Ka$, where K is a given positive number >1. Let $\Lambda(\Delta)$ be the volume of that portion of Δ lying inside the hyperboloid $ac - b^2 = 1$ and similarly for $\Lambda(\Sigma)$. (Thus if we regard the positive sheet of the hyperboloid $ac - b^2 = 1$ as a realization of the Lobatchevskian plane in the usual way, $\Lambda(\Sigma)$ is a constant times the Lobatchevskian area of that portion of the hyperboloid which lies in Σ.) If D is an odd positive integer let $h(-D)$ be the number of lattice points (a, b, c) on the hyperboloid $ac - b^2 = D$ such that a, $2b$, and c have g.c.d. 1 and (a, b, c) lies in Δ, and let $h_\Sigma(-D)$ denote the number of these lattice points which also lie in Σ. Thus, if to each triple (a, b, c) we make correspond the binary quadratic form $ax^2 + 2bxy + cy^2$, then $h(-D)$ is the number of reduced properly primitive binary quadratic forms of determinant $-D$. (Still another interpretation is in terms of 2 by 2 matrices L such that $L^2 = -DI$, where I is the 2 by 2 identity matrix.) By the Heilbronn-Siegel Theorem $h(-D)$ tends to infinity with D. The author proves that

if D goes to infinity through the odd integers such that $(-D|p)=1$ for some fixed odd prime number p, then

$$\lim_{D\to\infty} h_\Sigma(-D)/h(-D)=\Lambda(\Sigma)/\Lambda(\Delta).$$

This is an improvement of a result announced earlier [Dokl. Akad. Nauk SSSR (N.S.) 93 (1953), 973; MR 15, 856] to the effect that if K is sufficiently large, then $\liminf_{D\to\infty} h_\Sigma(-D)/h(-D)>0$ (with the same set of values for D). The arguments employed are related to those used in proving a similar result concerning the asymptotic distribution of lattice points on spheres [ibid. 96 (1954), 909–912; MR 16, 451 . A certain generalized quaternion algebra is an important tool. {The reviewer believes that in § 43 there is a minor error in a volume calculation which the author makes in applying the main theorem to a special situation.} *P. T. Bateman.*

Linnik, Yu. V. An new arithmetic application of the geometry of Lobačevskiĭ. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 112–114. (Ukrainian. Russian summary)

A summary of the paper reviewed above.
P. T. Bateman (Princeton, N.J.).

Rodeja F., E. G.-. On Fermat's last theorem. Las Ciencias **21** (1956), 382–383. (Spanish)

An error is pointed out in the proof offered in the article listed in MR **17**, 946.

Schinzel, André. Sur l'équation $\varphi(x)=m$. Elem. Math. **11** (1956), 75–78.

The question whether there is an infinitude of even integers m for which there equation $\varphi(x)=m$ has no solutions, is answered in the affirmative in this paper. For example, $\varphi(x)=2\cdot 7^k$ has solutions if and only if $k=0$, and then only three, viz. $x=3$, 4, 6. Related questions as to the form of m for which $\varphi(x)=m$ has exactly one, two or three solutions, are also considered. W. Sierpiński has made the conjecture that for every integer $s>1$ there exists an infinitude of integers m for which $\varphi(x)=m$ has precisely s solutions. It is not even known whether there exists a single m for which $\varphi(x)=m$ has precisely s solutions. *I. A. Barnett* (Cincinnati, Ohio).

Linnik, Yu. V. Markov chains in the analytical arithmetic of quaternions and matrices. Vestnik Leningrad. Univ. **11** (1956), no. 13, 63–68. (Russian)

A proof using less recherché machinery from the theory of probability of the author's result about primitive integral quaternions B which can be expressed in the form $R_1R_2\cdots R_s$, where each R_j is an integral quaternion of fixed norm r [Uspehi Mat. Nauk (N.S.) **9** (1954), no. 4(62), 203–210; MR **16**, 1002]. The present proof is also valid if B is a primitive integral 2 by 2 matrix and the R_j are taken from a fixed set of representatives of the classes of integral matrices of determinant r under right association. [Cf. the paper reviewed second above.]
J. W. S. Cassels (Cambridge, England).

See also: Kneser, p. 192.

Analytic Theory of Numbers

Selberg, Sigmund. Über eine zahlentheoretische Summe. Math. Scand. **4** (1956), 129–142.

In earlier papers [Norske Vid. Selsk. Forh., Trondheim **26** (1953), 89–93; **28** (1955), 37–41; MR **15**, 607; **17**, 348], the author proved that the sum

$$\sigma_k(x)=\sum_{n=1}^{k}\left\{x\left[\frac{r}{x}\right]-(x+1)\left[\frac{r}{x+1}\right]\right\}$$

satisfied the condition $\sigma_k(x)\geqq 0$ and that $\lim_{k\to\infty}\sigma_k(x)/k=\frac{1}{2}$. In this paper, he demonstrates that $\sigma_k(x)\leqq k$.
R. Bellman (Santa Monica, Calif.).

Newman, Morris. Generalizations of identities for the coefficients of certain modular forms. J. London Math. Soc. **31** (1956), 205–208.

Denoting the coefficient of x^n in the power series expansion of $\prod_{k=1}^{\infty}(1-x^k)^r$ by $P_r(n)$ (where n is a non-negative integer) and defining $P_r(n)$ to be zero otherwise the author proves for all integral n the identities:

$$(1)\quad P_r(nQ+\delta)=P_r(\delta)P_r(n)-p^{\frac{1}{2}r-1}P_r\left(\frac{\delta-\Delta}{p}\right)P_r\left(\frac{n-\Delta}{p}\right),$$

where r is even; $0<r\leqq 24$; p prime >3; $\Delta=r(p-1)/24$ is an integer; Q is a power of p; $\delta=r(Q-1)/24$.

$$(2)\quad P_r(nQ+\delta)=(-p)^{\frac{1}{2}r-1}P_r\left(\frac{\delta-\Delta}{p^2}\right)P_r\left(\frac{n}{p}\right),$$

where $r=2$, 4, 6, 8, 10, 14, 26; p is a prime >3 such that $r(p+1)\equiv 0 \pmod{24}$; $\Delta=r(p^2-1)/24$; Q is an odd power of p; $\delta=r(pQ-1)/24$. — These are generalizations of the case $Q=p$ which the author proved earlier [same J. **30** (1955), 488–493; MR **17**, 15]. *H. D. Kloosterman.*

Newman, Morris. On the existence of identities for the coefficients of certain modular forms. J. London Math. Soc. **31** (1956), 350–359.

Let p be a prime >3; r and K positive integers; Q_1, Q_2, $\cdots$, Q_K powers of p, either all squares or all non-squares; $\varepsilon=p$ or $=1$ according as the Q_k are squares or non-squares; $\nu_k=(pQ_k-\varepsilon)/24\varepsilon$ $(k=1, 2, \cdots, K)$;

$$\prod_{n=1}^{\infty}(1-x^n)^r=\sum_{n=0}^{\infty}p_r(n)x^n.$$

The author proves the existence of identities

$$\sum_{k=1}^{K} b_k p_r(Q_k n+r\nu_k)=b_0 p_r(\varepsilon n/p)$$

for all integers $n\geqq 0$ and sufficiently large K and where the b_k are not all zero. *H. D. Kloosterman* (Leiden).

Cohn, Harvey. Variational property of cusp forms. Trans. Amer. Math. Soc. **82** (1956), 117–127.

The author proves: A necessary and sufficient condition that a modular form $\varphi(z)$ of dimension -4 belonging to a subgroup G of the modular group be expressible as a linear combination of Eisenstein series with real coefficients is that a function $P(z)$ exists such that $\varphi(z)=P'''(z)$ and for which the function $q_T(z)$ defined by

$$q_T(z)=(cz+d)^2P(Tz)-P(z)$$

(for $T=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in G$) be a quadratic polynomial with real coefficients. *H. D. Kloosterman* (Leiden).

Dutta, Mahadeb. On new partition of numbers. Rend. Sem. Mat. Univ. Padova **25** (1956), 138–143.

Denote by $p_d(n)$ the number of partitions of n so that no summand occurs more than d times. Using a Tauberian theorem of Hardy and Ramanujan [Proc. London Math.

Soc. (2) 17 (1918), 75–115] the author proves that

$$\log p_d(n) = (1+o(1))\pi\sqrt{\left(\frac{2dn}{3(d+1)}\right)}.$$

He also obtains some identities for $p_d(n)$. *P. Erdős.*

Bateman, P. T.; and Erdös, P. Monotonicity of partition functions. Mathematika 3 (1956), 1–14.

Let $p(n)$ be the number of partitions of n into parts taken from the set A, repetitions being allowed; A is an arbitrary finite or infinite set of positive integers other than the empty set or the set consisting of the single element unity. The authors prove: $p(n)$ is strictly increasing for all large n if and only if the following condition is satisfied: A contains more than one element, and if any element is removed from A, the remaining elements have greatest common divisor unity. This result is important in applications of Ingham's Tauberian theorem for partitions.

The authors work with $p^{(k)}(n)$, which is the kth difference of $p(n)$ if $k>0$, the $(-k)$th order summatory function of $p(n)$ if $k<0$. In the course of the proof, they develop interesting information on the order of magnitude of $p^{(k)}(n)$. All arguments used are elementary, involving only formal power series and partial fraction decompositions of rational functions. *J. Lehner.*

Whiteman, Albert Leon. A sum connected with the series for the partition function. Pacific J. Math. 6 (1956), 159–176.

Let $p(n)$ denote the number of unrestricted partitions of n into positive integral parts. H. Rademacher's convergent modification of Hardy and Ramanujan's series for the partition function may be written

$$p(n) = \frac{2\sqrt{3}}{\mu(24n-1)}\sum_{k=-\infty}^{\infty}{}' \, |k|^{-\frac{1}{2}}A_k(n)(\mu-k)\exp(\mu/k),$$

where $\mu=\pi(24n-1)^{\frac{1}{2}}/6$. The coefficients $A_k(n)$ are the subject of the paper under review. They are defined by

$$A_{-k}(n) = A_k(n) = \sum_{\substack{1\le h\le k\\(h,k)=1}} \omega_{hk}\exp(-2\pi ihn/k),$$

whence ω_{hk} are certain complicated $24k$th roots of unity arising from the theory of Dedekind's modular function $\eta(\tau)$. In 1936 the reviewer discovered that, despite its complicated definition, $A_k(n)$ is a multiplicative function of k and that when k is a power of a prime $k^{-\frac{1}{2}}A_k(n)$ is simply twice the cosine of a rational multiple of π. The resulting simple formulas for $A_k(n)$ made possible the use of (1) for calculating $p(n)$ for very large isolated values of n.

Later Rademacher and Whiteman derived these results from the theory of Dedekind sums $S(h, k)$ [Amer. J. Math. 63 (1941), 377–407; Bull. Amer. Math. Soc. 53 (1947), 598–603; MR 2, 249, 8, 567]. In the present paper the results are obtained once more; this time using the following finite Fourier series for $A_k(n)$

$$3^{\frac{1}{2}}k^{-\frac{1}{2}}Ak(n) = \sum (-1)^r\cos(\pi(6r+1)/(6k))$$

the sum extending over all r mod k for which

$$(3r^2+r)/2 \equiv -n \pmod{k}.$$

This unpublished result of A. Selberg is shown to be equivalent to the above definition. The author uses a sum deu to W. Fisher [Pacific J. Math. 1 (1951), 83–95; MR 13, 209] to carry out the evaluation of $A_k(n)$ when k is a power of a prime. *D. H. Lehmer* (Berkeley, Calif.).

★**Niven, Ivan. Irrational numbers.** The Carus Mathematical Monographs, No. 11. The Mathematical Association of America. Distributed by John Wiley and Sons, Inc., New York, N.Y., 1956. xii+164 pp. $3.00.

This book certainly fulfils the requirements of a Carus monograph in that the exposition is comprehensible to a wide group of readers. It may be read with profit by those having a moderate acquaintance with hardly more than elementary mathematics, as well as by the trained mathematician. Considerable emphasis is naturally placed on diophantine approximation, although this part of the work is self-contained. While most of the results of the book are well-known, the treatment is fresh and lively. The practice of giving historical notes at the end of each chapter, so well utilized by Hardy and Wright in their "Theory of Numbers" [3rd ed., Oxford, 1954; MR 16, 673] is especially interesting and helpful.

The book opens with the usual remarks on rational and irrational numbers and a proof of Cantor's criterion for the decimal expansion of a real number. The author wisely refrains from giving the construction of the real numbers from the rationals. Next, he establishes the irrationality of the trigonometric functions for non-zero rational values of the arguments. He then considers the trigonometric functions for arguments which are rational multiples of π, and proves a result due to D. H. Lehmer that $2\cos 2\pi kn^{-1}$ is an algebraic integer $(n>2)$.

In Chapter 4 the author begins with the simplest notions of diophantine approximation, which in Chapter 6 culminates in the result due to A. Hurwitz that for any irrational number ξ there exist infinitely many rationals h/k such that $|\xi-h/k|<1/5^{1/2}k^2$, where $5^{1/2}$ is the best possible constant. Chapter 7, on algebraic and transcendental numbers, deals with orders of approximation, and it is proved that a real algebraic number of degree n is not approximable to order $n+1$ or higher. As a consequence of this result one may readily prove that all Liouville numbers are transcendental.

The author devotes Chapter 8 to the study of properties of the normal numbers of E. Borel. The subject is treated here more fully than in Hardy and Wright. In Chapter 9 the work of Lindemann on the transcendence of π, with extensions and generalizations, is considered. The principal result is that if $\alpha_1, \alpha_2, \cdots, \alpha_m$, are any distinct algebraic numbers the equation $\sum_{j=1}^m a_je^{\alpha_j}=0$ is impossible in algebraic numbers. From this result one obtains not only the transcendence of π and e, but also of e^α, $\sin\alpha$, $\sinh\alpha$ etc., for any non-zero algebraic number α. Here the reviewer feels that the author's own elegant proof of the transcendence of π should have been included. The book closes with a clear and interesting account of the Gel'fond-Schneider theorem, viz., if α and β are algebraic numbers $(\alpha\neq 0,1)$ and if β is not a real rational number, every value of α^β is transcendental.

The author is to be congratulated on bringing together, in a compact little book of about 150 pages, a clear and concise treatment of some of the more interesting properties of irrational and transcendental numbers.

I. A. Barnett (Cincinnati, Ohio).

Theory of Algebraic Numbers

Gundlach, Karl-Bernhard. Poincarésche und Eisensteinsche Reihen zur Hilbertschen Modulgruppe. Math. Z. 64 (1956), 339–352.

In a previous paper, the author constructed a Hecke-

Petersson theory for the Hilbert modular group [Acta Math. 92 (1954), 309–345; MR 16, 1000]. This is a hyperabelian substitution group based on a totally real algebraic field K of degree n in which the complex variable vector $(\tau^{(1)}, \cdots, \tau^{(n)})$ is restricted to lie in the region $\operatorname{Im} \tau^{(j)}>0$, $j=1, \cdots, n$. In the present paper, the author points out that one may just as well consider the 2^n groups obtained by allowing the $\tau^{(j)}$ to have an arbitrary combination of signs, since in each case, the domain of the vector is mapped into itself by all the transformations of the group.

Moreover, O. Herrmann [Math. Ann. 127 (1954), 357–400; MR 15, 940] has discussed besides the Hilbert group, a whole series of related groups corresponding to the ideal classes of K. In the present paper, the author treats the Poincaré and Eisenstein series in the related groups $\Gamma_1(K_c)$ (c an ideal in K) in all of the 2^n regions. He obtains results similar to those in his first paper: the Poincaré series form a basis for the entire cusp forms; the Fourier coefficients of the entire cusp-forms have the estimate of the previous paper; and the Eisenstein series span the orthogonal complement (in the space of entire forms) of the cusp forms. These results hold for $r\geqq 2$ ($-r$ is the dimension of the modular form). The last section treats the linear independence of the Eisenstein series for $r=1$. *J. Lehner* (Los Alamos, N.M.).

Cohn, Harvey. Numerical study of signature rank of cubic cyclotomic units. Math. Tables Aids Comput. 8 (1954), 186–188.

A description is given of a special calculation in the theory of cubic fields, more particularly the fields generated by the Gaussian cubic sums

$$\sum_{x=0}^{p-1} \exp(2\pi i x^3/p),$$

where the p's are primes of the form $6n+1$. The problem was to determine those p's for which the three cyclotomic units

$$\theta_i = -\prod_{t=0}^{n-1} \sin(\tfrac{1}{2}\pi g^{3t+i}/p) \csc(\tfrac{1}{2}\pi g^{3t+i+1}/p) \quad (i=0, 1, 2)$$

are all positive. Here g is an odd primitive root of p. Instead of trying the rather hopeless task calculating θ_i approximately, the sign of θ_i can obviously be found exactly by tallying the number of times that the above powers of g modulo $2p$ lie in the interval $(0, p)$. The calculation was performed for the 207 values of p under 3000 in about an hour on the MIDAC. In only 21 cases (163, $\cdots$, 2803) the θ_i's are all positive; the "expected" number is 52. *D. H. Lehmer* (Berkeley, Calif.).

Mostowski, A. Determination of the degree of certain algebraic numbers. Prace Mat. 1 (1955), 239–252. (Polish. Russian and English summaries)

Soient W le corps de nombres rationnels, W^* son groupe multiplicatif, W^{*p} le groupe des puissance p-ièmes des éléments de W^*, $D_1, D_2, \cdots, D_s$ des éléments entiers de W^* incongrûs deux à deux (mod W^{*p}). Soit r la dimension du p-groupe abélien élémentaire

$$W^{*p}(D_1, D_2, \cdots, D_s)/W^{*p},$$

considéré comme un espace véctoriel sur le corps Γ_p de p éléments. Si $\bar{a}$ désigne la classe (mod p) d'un entier rationnel a, et si $D_i=\prod_j p_j^{a_{ij}}$ est la décomposition de D_i en facteurs premiers, r est manifestement égal au rang de la matrice $(\bar{a}_{ij})$ et est introduit par l'auteur sous cette forme. L'auteur démontre, en employant la théorie de Galois et à titre d'une illustration simple de cette théorie que, pour tous $c_1, c_2, \cdots, c_s \in W^*$, le degré de $c_1D_1^{1/p}+c_2D_2^{1/p}+\cdots+c_sD_s^{1/p}$ par rapport à W est p^r. L'auteur n'affirme pas la nouveauté de ce résultat et dit ne pas avoir cherché s'il n'avait pas été déjà publié quelque part. Le referent l'ignore également, mais a le sentiment que ce résultat est connu par la quasi-totalité des algébristes. Il est, d'ailleurs, exact quand W est un corps quelconque et les $D_i \in W^*$ ne sont assujetis qu'à la seule condition d'être incongrûs deux à deux (mod W^{*p}), et, si l'on n'a pas en vue d'illustrer la théorie de Galois, il peut se démontrer plus simplement. *M. Krasner* (Paris).

See also: Nakayama, p. 191; Klingen, p. 191; Kneser, p. 192.

Geometry of Numbers

Mahler, K. Invariant matrices and the geometry of numbers. Proc. Roy. Soc. Edinburgh. Sect. A. 64 (1956), 223–238.

This is a generalization of earlier results [Proc. London Math. Soc. (3) 5 (1955), 358–379; MR 17, 589].

Let $R_n=\{X\}$ denote affine n-space with the unit vectors $U_1=(1, 0, \cdots, 0)$, $U_n=(0, \cdots, 0, 1)$. Given a p-linear mapping

$$M: X^{(1)}, \cdots, X^{(p)} \rightarrow [X^{(1)}, \cdots, X^{(p)}]$$

of the p-tuples of R_n into affine N-space R_N with the following properties: (i) There are N sets of indices $\nu_{\eta 1}, \cdots, \nu_{\eta p}$ such that the points $Y_\eta=[U_{\nu_{\eta 1}}, \cdots, U_{\nu_{\eta p}}]$ span R_N ($\eta=1, 2, \cdots, N$). (ii) To every non-singular affine transformation T of R_n there is an affine transformation T^* of R_N such that $[TX^{(1)}, \cdots, TX^{(p)}]=T^*[X^{(1)}, \cdots, X^{(p)}]$. (iii) There is a constant P such that $|\det T^*|=|\det T|^P$ for all T. (iv) M maps the p-tuples of rational points onto rational points. — If $T=T(t_1, \cdots, t_n)$ is the transformation $TU_\lambda=t_\lambda U_\lambda$, then

$$T^*Y_\eta=t_1^{\alpha_{\eta 1}} \cdots t_n^{\alpha_{\eta n}} \cdot Y_\eta,$$

where the $\alpha_{\eta\nu}$ are non-negative integers with $\alpha_{\eta 1}+\cdots+\alpha_{\eta n}=p$. Put $q=q(M)=\max_{\eta,\nu}\alpha_{\eta\nu}$. — The positive constants $c_1, c_2, \cdots$ depend only on M.

A "convex body" is a closed bounded convex set with interior points which is symmetric with respect to the origin. Theorem 1: If $X^{(1)}, \cdots, X^{(p)}$ range through the convex body K in R_n, the convex hull of the points $[X^{(1)}, \cdots, X^{(p)}]$ is the associated convex body K. Let $V(K), \cdots$ denote the volume of $K \cdots$. Theorem 3: There are c_1, c_2 such that $c_1V(K)^p \leqq V(\mathrm{K}) \leqq c_2V(K)^p$. — Let $F(X)$ and $\Phi(\Xi)$ denote the distance functions of K and K respectively. Theorem 4:

$$\Phi([X^{(1)}, \cdots, X^{(p)}]) \leqq F(X^{(1)}) \cdots F(X^{(p)})$$

for all $X^{(\pi)} \subset R_n$. — Let $L_0(\Lambda_0)$ be the lattice of the integral points of R_n (of R_N). Let $0<m_1\leqq m_2 \cdots \leqq m_n$ ($0<\mu_1 \leqq \cdots \leqq \mu_N$) be the successive minima of K in L_0 (of K in Λ_0). Let $M_1, \cdots, M_N$ denote the N products $m_{\nu_{\eta 1}} \cdots m_{\nu_{\eta p}}$ arranged in order of increasing size. Theorem 5: $c_3M_\eta \leqq \mu_\eta \leqq c_4M_\eta$ for some c_3, c_4 ($\eta=1, \cdots, N$). Theorem 6: $m_1^p \leqq c_5\mu_1$, $\mu_1^{n-1} \leqq c_6V(K)^{q-p}m_1^{nq-p}$ for some c_5, c_6. — Let $T=(a_{hk})$ be a unimodular affine transformation; $T^*=(\alpha_{\eta\kappa})$. They define distance functions

$$F(X)=\max_{h=1,\cdots,n} \left|\sum_{k=1}^n a_{hk}x_k\right|, \quad \Phi(\Xi)=\max_{\eta=1,\cdots,N} \left|\sum_{\kappa=1}^N \alpha_{\eta\kappa}\xi_\kappa\right|$$

of the points $X=(x_1, \cdots, x_n)\subset R_n$ and $\Xi=(\xi_1, \cdots, \xi_N)\subset R_N$. Put $m_1=\min_{X\subset L_0} F(X)$, $\mu_1=\min_{\Xi\subset\Lambda_0}\Phi(\Xi)$. Theorem 6 leads to Theorem 7: $m_1{}^p\leqq c_1\mu_1$; $\mu_1{}^{n-1}\leqq c_8 m_1{}^{nq-p}$ [cf. Mahler, Czechoslovak Math. J. 5(80) (1955), 180–193; MR 17, 589]. — The paper is concluded "with some short remarks on the connections to representation theory."

P. Scherk (Philadelphia, Pa.).

See also: Linnik, p. 193.

ANALYSIS

Functions of Real Variables

Geffroy, J. Sur une relation entre les dérivées partielles f_{xy}'' et f_{yx}''. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 533–542.

By making use of the Lebesgue integral theory the author establishes necessary and sufficient conditions for the equality of the second partial derivatives, $f_{xy}''(x, y)$, $f_{yx}''(x, y)$. If $f(x, y)$, f_{xy}'', f_{yx}'' are defined in a domain D, and if f_{xy}'' is bounded in this domain, the necessary and sufficient conditions that $f_{yx}''=f_{xy}''$ is that f_{xy}'', considered as a function of x, be a derived function. The condition necessary and sufficient that $f_{xy}''(x_0, y_0)=f_{yx}''(x_0, y_0)$ is that $f_{xy}''(x, y_0)$, considered as a function of x, be a derived function. These results are illustrated by examples.

R. L. Jeffery (Kingston, Ont.).

Kolmogorov, A. N. On the representation of continuous functions of several variables by superpositions of continuous functions of a smaller number of variables. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 179–182. (Russian)

Let E^r denote the n-dimensional unit cube $0\leqq x_i\leqq 1$ ($i=1, 2, \cdots, n$). The main result of the paper is as follows. 1) For any $n\geqq 3$ there exist on E^{n-1} continuous real-valued functions

$$g_1{}^1, g_1{}^2, \cdots, g_1{}^n;\ g_2{}^1, g_2{}^2, \cdots, g_2{}^r$$

such that each f defined and continuous on E^n can be written

$$f(x_1, \cdots x_n) = \sum_{r=1}^{n} h_f{}^r[x_n, g_1{}^r(x_1, \cdots, x_{n-1}), g_2{}^r(x_1, \cdots, x_{n-1})],$$

where the $h_f{}^r$ are given and continuous on E^3. This result is a consequence of the following two theorems, the second of which is, in turn, a corollary of the first. 2) Let Ξ be a universal tree (by a tree we mean a locally connected continuum without subsets homeomorphic to a circumference; Ξ is a universal tree if it contains homeomorphic images of all trees); then for each $n\geqq 2$ there exist on E^n continuous functions $\varphi^1, \varphi^2, \cdots, \varphi^{n+1}$ with values from Ξ, such that every continuous real-valued f defined on E^1 is representable in the form

$$f(x_1, \cdots, x_n)=\sum_{r=1}^{n+1} h_f{}^r[\varphi^r(x_1, \cdots, x_n)\ ,$$

where the $h_f{}^r(\xi)$ are defined and continuous for $\xi\in\Xi$; the functions $h_f{}^r$ can be chosen so that they depend continuously on f, in the topology of uniform convergence in the spaces of continuous functions on E^n and Ξ. 3) For each $n\geqq 3$ there exist on E^n continuous functions $\varphi^1, \cdots, \varphi^n$ with values in Ξ, so that every function f defined and continuous on E^n can be written

$$f(x_1, \cdots, x_n)=\sum_{r=1}^{n} h^r[x_n, \varphi^r(x_1, \cdots, x_{n-1})\ ,$$

where the real-valued functions $h^r(x, \xi)$ are defined and continuous on the product $E^1\times\Xi$. The idea of the proof is indicated. In the beginning of the paper Kolmogorov says that from the above stated main result follows "a somewhat unexpected corollary: an arbitrary continuous function of no matter how many variables may be represented as a finite superposition of continuous functions of not more than three variables."

A. Zygmund.

Prékopa, A. Extension of multiplicative set functions with values in a Banach algebra. Acta Math. Acad. Sci. Hungar. 7 (1956), 201–213. (Russian summary)

Let $\mathfrak{R}$ be a ring of subsets of a given set X and let $\mathfrak{S}(\mathfrak{R})$ be the smallest σ-ring of subsets of X, containing $\mathfrak{R}$. Let $\mathfrak{B}$ be a commutative Banach algebra with unity e. A set function $f(A)$ with domain $\mathfrak{R}$ and range in $\mathfrak{B}$ is called multiplicative (completely multiplicative) in $\mathfrak{R}$ if for every pair of disjoint sets $A_1, A_2\in\mathfrak{R}$ (for every sequence of disjoint sets $A_k\in\mathfrak{R}$ with their union A in $\mathfrak{R}$) the relation $f(A_1+A_2)=f(A_1)\cdot f(A_2)$ ($f(A)=\prod_{k=1}^{\infty} f(A_k)$) holds. The author proves the following extension theorem 1: Let $f(A)$ be a completely multiplicative set function defined on $\mathfrak{R}$ with $\|f(A)\|\leqq 1$ ($A\in\mathfrak{R}$). If for every sequence $\{A_k\}$ of disjoint sets of $\mathfrak{R}$ the relation $\sum_{k=1}^{\infty}\|e-f(A_k)\|<\infty$ holds, then there is one and only one completely multiplicative set function $f^*(A)$ defined on $\mathfrak{S}(\mathfrak{R})$, for which $f^*(A)=f(A)$ when $A\in\mathfrak{R}$. If $\{A_k\}$ is a convergent sequence of sets of $\mathfrak{S}(\mathfrak{R})$ with limit A, then $\lim_{k\to\infty} f^*(A_k)=f^*(A)$. The author proves also a quite analogous, but more general theorem 2 where the condition $\|f(A)\|\leqq 1$ ($A\in\mathfrak{R}$) is replaced by the condition that there is a bounded, completely additive, real-valued set function $\varphi(A)$ defined on $\mathfrak{R}$, for which $\|f(A)\|\leqq 2^{\varphi(A)}$ ($A\in\mathfrak{R}$).

A. Rosenthal (Lafayette, Ind.).

Jerison, M.; and Rabson, G. Convergence theorems obtained from induced homomorphisms of a group algebra. Ann. of Math. (2) 63 (1956), 176–190.

Let G be a locally compact group, K_1 a compact normal subgroup with normalized Haar measure, and f a measurable function on G which is summable on sets of finite measure .Then the mapping $f\to f_1$, where $f_1(x)=\int_{K_1} f(xy)dy$ preserves all the algebraic properties of the standard function spaces on the groups G and G/K_1. The authors prove that if $\{K_n\}$ is a sequence of compact normal subgroups decreasing to the group identity, then the corresponding functions f_n converge to f in various ways. In particular, under slightly stronger assumptions on f, f_n converges to f pointwise almost everywhere. In conclusion, the authors subsume under these results various known expansion theories, such as that of the Walsh-Rademacher functions and their generalizations.

L. H. Loomis (Cambridge, Mass.).

See also: Fuglede, p. 198; Besov, p. 202; Faleschini, p. 203; Volpato, p. 212; Bakel'man, p. 230; Pinsker, p. 239; Volt, p. 256.

Measure, Integration

LeBlanc, L.; and Fox, G. E. On the extension of measure by the method of Borel. Canad. J. Math. **8** (1956), 516–523.

The paper gives a "direct" proof of the Hahn-Kolmogoroff extension theorem. The proof is direct in the sense that it does not proceed (à la Caratheodory) via outer measures, but goes directly (via transfinite induction) from a ring (of sets) to the generated σ-ring.

P. R. Halmos (Chicago, Ill.).

Fuglede, Bent. On a theorem of F. Riesz. Math. Scand. **3** (1955), 283–302 (1956).

Let X be a set and $\mathfrak{M}$ a σ-algebra of subsets of X. Let $\mathfrak{A}$ be a subfamily of $\mathfrak{M}$ such that the void set and all finite disjoint unions of sets in $\mathfrak{A}$ form an algebra $\mathfrak{R}$ of sets, and such that the smallest σ-algebra containing $\mathfrak{R}$ is $\mathfrak{M}$. Let μ be a non-negative, countably additive, extended-real valued measure on $\mathfrak{M}$ such that $0<\mu(A)<+\infty$ for all $A \in \mathfrak{A}$. Let ϕ be a complex-valued, finitely additive set-function defined on $\mathfrak{A}$. Let p be a real number, $1<p<+\infty$. Theorem. There is a function $f \in L^p(X, \mathfrak{M}, \mu)$ such that

$$\phi(A)=\int_A f(x)d\mu(x), \text{ for all } A \in \mathfrak{A},$$

if and only if there is a positive real number c such that

$$\sum_{j=1}^{n}|\phi(A_j)|^p(\mu(A_j))^{1-p}\leq c$$

holds for every finite family $A_1, \cdots, A_n$ of pairwise disjoint sets in $\mathfrak{A}$. The function f is then uniquely determined as an element of $L^p(S, \mathfrak{M}, \mu)$ and the least possible value of c is $\|f\|_p{}^p$. Analogous theorems are proved for $L^1(X, \mathfrak{M}, \mu)$ and for $L^\infty(X, \mathfrak{M})$. (These latter theorems are actually special cases of the Radon-Nikodym theorem.) The first theorem is a generalization to the abstract situation of a classical result of F. Riesz [Math. Ann. **69** (1910), 449–497]. It is also a modest generalization of a theorem of Fullerton [Proc. Amer. Math. Soc. **5** (1954), 689–696; MR **16**, 263]. The author also gives generalizations of his results to the case of functions whose values lie in k-dimensional Euclidean spaces and to functions $F(t)$ $(t\geq 0)$ that generalize the functions $|t|^p$.

E. Hewitt (Seattle, Wash.).

Climescu, Al. Une application du théorème de Weierstrass-Bernstein dans le calcul intégral. Bul. Inst. Politehn. Iaşi (N.S.) **2** (1956), 9–11. (Romanian. Russian and French summaries)

On démontre l'existence d'une primitive d'une fonction continue dans un intervalle fermé en partant du théorème de Weierstrass-Bernstein et on indique les avantages que présente l'introduction d'une telle démonstration dans l'enseignement.

Résumé de l'auteur.

Volpato, Mario. Sulla formula di Green nell'ambito delle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 30–37.

The author proves the formula

$$\int_{\varphi(c)}^{\varphi(y)}f(t,y)dt=\int_c^y f[\varphi(t),\ t]\varphi'(t)dt+\int_c^y d\eta\int_{\varphi(c)}^{\varphi(\eta)}f_y'(t,\ \eta)dt,$$

assuming that: I) $\varphi(y)$ is defined and absolutely continuous in J $(c\leq y\leq d)$, and $\varphi(y) \in I(a\leq x\leq b)$; II) $f(x, y)$ is defined in $R=I\times J$, $f(x, y)$ is measurable in x, absolutely continuous with respect to y, for almost all $x \in I$, and $|f(x, y)|\leq M(x)$, where $M(x)$ is integrable in I; III) $f[\varphi(y), y]\varphi'(y)$ is integrable in J; IV) $f_y'(x, y)$ exists almost everywhere and is integrable in R.

M. Cotlar.

Haupt, Otto; und Pauc, Christian Y. Bemerkungen über Unterteilungsintegrale und lineare Funktionale. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1955**, 347–369 (1956).

Let G be a set, $\mathfrak{q}$ a ring of subsets of G, and i a real-valued, non-negative content (=finitely additive set function) on $\mathfrak{q}$. The authors give a careful definition of upper and lower integrals with respect to i for all real-valued functions f on G such that $\{x: x \in G, f(x)\neq 0\}\subset M$ for some $M \in \mathfrak{q}$. This definition is similar to the usual definition for Riemann integrals. The first theorem shows how to write the upper and lower integrals in terms of certain integrals over real intervals $[\beta, \gamma]$. The present definition is compared with definitions given by Ridder [Fund. Math. **24** (1935), 72–117], Loomis [Amer. J. Math. **76** (1954), 168–182; MR **15**, 631], and the reviewer [Ark. Mat. **2** (1952), 269–282; MR **14**, 545]. Various extensions of the integral and the content i are also constructed.

E. Hewitt (Seattle, Wash.).

Swift, George. n-valued irregular Borel measures. Duke Math. J. **23** (1956), 393–407.

Terminology and notations are taken from a previous paper by the author [Duke Math. J. **22** (1955), 427–433; MR **17**, 245]. X denotes a topological space. The property for the one-point sets to be compact, hence Borel, is frequently used. The contents of § 2 can be exemplified by Theorem 2.10: Let μ be a totally finite, n-valued, irregular Borel measure on X. Then $\mu=\varphi+\sum \mu_i$ $(i=1, \cdots, m)$, where φ is a totally finite, p-valued regular measure on X, where p is an integer $(1\leq p<n)$, and each μ_i is a totally finite, 2-valued, irregular Borel measure on X, where m is an integer $(1\leq m<n)$. μ is outer [inner] irregular at a Borel set B if and only if there exists an i_0 $(i_0=1, \cdots, m)$ such that μ_{i_0} is outer [inner] irregular at B. In § 3 necessary and sufficient conditions on X are given for the existence of a totally finite n-valued irregular Borel measure. To such a measure is attached a class $\mathcal{A}$ of Borel sets possessing equivalent properties $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ such that the value of the measure at any Borel set may be computed by means of the values of the measure on this class and on the compact sets of the space. Here is $\mathcal{P}_1$: (1) $A_1, A_2 \in \mathcal{A}$ imply the existence of $A \in \mathcal{A}$ such that $A\subset A_1\cap A_2$, and (2) $A \in \mathcal{A}$ and $A=\bigcup B_i$ $(i=1, 2, \cdots)$, where the B_i are pairwise disjoint Borel sets, imply that there exist an i_0 and an $A_0 \in \mathcal{A}$ such that $A_0\subset B_{i_0}$. For a totally finite, 2-valued Borel measure ν on X, the class of the Borel sets A such that $\nu(A)>0$ satisfies $\mathcal{P}_1$. To each μ_i in Theorem 2.10 is attached a class $\mathcal{A}_i$ of Borel sets with $\mathcal{P}_1$ such that $\mu_i(E)=c_i>0$ when there exists $A \in \mathcal{A}_i$, where $A\subset E$, $\mu_i(E)=0$ otherwise. If E is a Borel set and if for every open Borel set V [compact set C] such that $V\supset E$ $[C\subset E]$ there exists an $A \in \mathcal{A}_i$ such that $A\subset V-E[A\subset E-C]$, then μ_i is outer [inner] irregular at E.

C. Pauc (Lafayette, Ind.).

Silverman, E. An intrinsic inequality for Lebesgue area. Pacific J. Math. **6** (1956), 363–372.

If Q denotes the square $0\leq u,v\leq 1$, and a, b, c, d the four consecutive sides of Q, if x denotes any continuous mapping from Q into the space m of the bounded sequences, i.e., a surface S in m, let $L(x)$ be the Lebesgue area of x, i.e., of the surface S, let $x=lx_n$ be any monotone-

light factorization of x, and $M = x_\mu(Q)$ be the middle space. For any pair p, q of points $p, q \in Q$ we shall consider all curves g in M joining $x(p)$, $x(q)$; i.e., all continuous mappings $g = g(t)$, $0 \leq t \leq 1$, from $[0, 1]$ into M with $g(0) = x_\mu(p)$, $g(1) = x_\mu(q)$. Then lg is a continuous curve in S and the infimum, say $\hat{x}_G(p, q)$ of the lengths of the curves lg for all g above is a pseudo geodesic distance of the points $x(p)$, $x(q)$ on S. Then by $A(x)$ $[B(x)]$ is denoted the infimum of $\hat{x}_G(p, q)$ for all $p \in a$, $q \in c$ $[p \in b, q \in d]$. One of the main inequalities proved by the author is that $L(x) \geq A(x)B(x)$.

The author then defines various functionals having the character of an area. For instance if we consider all finite partitions H of Q into non-overlapping sets h whose boundaries can be divided into four consecutive "sides" as for Q, then an area $\Phi(x)$ is obtained as the supremum for all H of the sums $\sum A_h(x)B_h(x)$, where $A_h(x)$, $B_h(x)$ are defined for h as A, B for Q. The author then proves the main equality $L(x) = \Phi(x)$ for all x. The proofs draw upon previous papers of the author [Riv. Mat. Univ. Parma 2 (1951), 47–76, 195–201; MR 13, 122, 731] and on some results of the reviewer [Bull. Amer. Math. Soc. 59 (1953), 534; Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 13 (1943), 1323–1481; MR 8, 142]. *L. Cesari.*

See also: Geffroy, p. 197; Matsushita, p. 203; Ishii, p. 225; Alda, p. 241; Simonenko, p. 265; Choquet, p. 219.

Functions of Complex Variables,

★**Еругин, Н.П. Неявные функции. [Erugin, N. P. Implicit functions.]** Izdat. Leningrad. Univ., Leningrad, 1956. 59 pp. 2 rubles.

The booklet deals with implicit functions, mostly in the complex domain, given by one equation $F(x, y) = 0$ or by two $\phi_i(x, y, z) = 0$ $(i = 1, 2)$. The most elementary theorems are taken for granted. Weierstrass' preparation theorem is proved. Estimates for the radii of convergence of the series representing the implicit functions in case F or ϕ_1 and ϕ_2 are analytic are given. The behavior of the implicit functions under analytic continuation is discussed. *H. Busemann* (Los Angeles, Calif.).

Zimmermann, Eduard. Quasikonforme schlichte Abbildungen im dreidimensionalen Raum. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg 5 (1955/56), 109–115.

Quasiconformal mappings in three dimensions are here defined in terms of the ratio of the greatest and smallest axis of the ellipsoid that corresponds to an infinitesimal sphere. Lower bounds for the maximal dilatation are obtained for mappings of a rectangular box on another when the edges correspond, and a similar result is obtained for the mapping of a hollow box on another. The methods are very close to the original methods of Grötzsch. *L. V. Ahlfors* (Cambridge, Mass.).

Reade, Maxwell O. On Newtonian vector functions. Math. Scand. 4 (1956), 153–156.

Newtonian vectors, or vectors for which both the curl and the divergence vanish identically, can be considered as n-dimensional generalizations of analytic functions of a complex variable [Fulton and Rainich, Amer. J. Math. 54 (1932), 235–241; Beckenbach, Bull. Amer. Math. Soc. 48 (1942) 937–941; MR 4, 135]. In the present note, results of Fédoroff [Mat. Sb. 41 (1934), 92–98] and the author [Bull. Amer. Math. Soc. 55 (1949), 289] in complex-variable theory are extended to this more general case. Thus it is shown that, in the class of continuous vectors $V(x_1, x_2, x_3)$ in a domain D, Newtonian vectors are characterized by the equations

$$\lim_{k\to\infty} \frac{1}{r_k^5} \iiint_{|x-x_k|\leq r_k} (x - x_k) \cdot V(x) dv(x) = 0,$$

$$\lim_{k\to\infty} \frac{1}{r_k^5} \iiint_{|x-x_k|\leq r_k} (x - x_k) \times V(x) dv(x) = 0,$$

for a fixed null sequence $\{r_k\}$ of positive numbers and for every sequence $\{x_k\}$ in D that converges to a point of D. *E. F. Beckenbach* (Los Angeles, Calif.).

Rizza, G. B. Sulle condizioni di regolarità delle funzioni in un'algebra. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 38–43.

Soient $A = R^n$ une algèbre réelle d'ordre n, $[u_h] = [u_1, \cdots, u_n]$ une base de A et $\gamma_{hk}{}^r$ les correspondantes constantes de multiplication. Une fonction $y = \sum_k y^k u_k$ de la variable $x = \sum_h x^h u_h$ est régulière à droite par rapport à $[u_h]$, si, et seulement si, on a $\sum_h \partial y / \partial x^h u_h = 0$. Soit $\|a^h{}_{h'}\|$ la matrice définissant le changement de base $[u_h] \to [u_{h'}]$. L'auteur démontre le résultat suivant: si A est munie d'élément unité et que l'on désigne par P la matrice $\|P^{hr}\|$ où $P^{hr} = \sum_{h'} a_{k'}{}^h a_{k'}{}^r$ et par X la matrice correspondante à l'élément générique x de A dans la première représentation de A, alors une condition nécessaire et suffisante pour que toute fonction régulière à droite par rapport à $[u_h]$ soit encore régulière à droite par rapport à $[u_{h'}]$ (et réciproquement) est que l'on ait identiquement $XP = PX$. Ce résultat est généralisé, sous une forme un peu différente, au cas où A ne contient aucun élément $\nu \neq 0$ tel que $x\nu = 0$ pour tout $x \in A$. *J. Sebastião e Silva* (Lisbonne).

Batyrev, A. V. On singular integrals in a complex region. Rostov. Gos. Ped. Inst. Uč. Zap. 1953, no. 2, 39–47. (Russian)

Let $f(z)$ be holomorphic in $|z| < 1$, continuous in $|z| \leq 1$. By using the representation

$$f_n(z) = \frac{1}{2\pi i} \int_{|t|=1} f(t) \frac{t^n - z^n}{t - z} \cdot \frac{dt}{t^n}$$

for the partial sums of the Taylor series of $f(z)$, the author obtains a sufficient condition under which $f_n(z) \to f(z)$ uniformly in $|z| \leq 1$. {The result, however, is an immediate consequence of the Dini-Lipschitz test for Fourier series.} Applications are made to conformal mapping, but the results obtained are only special cases of known results. Another application is made to Faber polynomials: If the region D possesses a boundary curve with bounded curvature, then the Faber polynomials for the region form a basis for the set of all functions holomorphic in D and continuous in $\bar{D}$. Next, the author considers a similar integral representation for the first Cesàro means of the Taylor series of $f(z)$, but the result obtained is a direct consequence of Fejér's classical theorem for Fourier series. *W. Seidel* (Notre Dame, Ind.).

Čemeris, V. S. On the behavior of the derivative on the boundary of image regions. Dopovidi Akad. Nauk Ukrain. RSR 1955, 425–428. (Ukrainian. Russian summary)

Let $\tilde{D}$ be a simply connected region interior to, but not

identical with, $|\omega|<1$, which has $\omega=0$ as interior point and let the boundary of $\tilde{D}$ contain a free arc α_ω on $|\omega|=1$. Let the unit circle $|z|<1$ be mapped conformally onto $\tilde{D}$ so that $z=0$ is carried into $\omega=0$, and let the arc α_ω correspond to the arc α_z on $|z|=1$ under this mapping. The author obtains an estimate for the modulus of the derivative of the mapping function on α_z and shows, in particular, that it is always strictly greater than 1 at all interior points of α_z. *W. Seidel* (Notre Dame, Ind.).

Tsuji, Masatsugu. A simple proof of Bieberbach-Grunsky's theorem. Comment. Math. Univ. St. Paul. **5** (1956), 29–32.

A new proof of the classical result that there exists a $(1, n)$ conformal mapping of a plane domain D of connectivity n onto the unit circle which carries a given point on each of the boundary components of D into the same point of the unit circumference. *Z. Nehari.*

Ozawa, Mitsuru. Some estimations on the Szegö kernel function. Kōdai Math. Sem. Rep. **8** (1956), 71–78.

The paper presents a variety of results, both old and new, concerning the Szegö kernel function $K(z, \bar{z}_0)$ of a plane domain B bounded by n curves Γ. A typical estimate of the type treated here is the not unfamiliar inequality $K(z_0, \bar{z}_0)\leqq(\frac{1}{4}\pi^2)\int_\Gamma |dz|/|z-z_0|^2$. *P. R. Garabedian* (Stanford, Calif.).

Kusunoki, Yukio. Some classes of Riemann surfaces characterized by the extremal length. Proc. Japan Acad. **32** (1956), 406–408.

Announcement of results concerning the characterization of parabolic Riemann surfaces in terms of extremal length, the boundary behavior of bounded harmonic functions on a neighborhood of the ideal boundary of Riemann surfaces belonging to a certain subclass of O_G and to the Riemann relations for such surfaces. *M. Heins* (Princeton, N.J.).

Boas, R. P., Jr. Interference phenomena for entire functions. Michigan Math. J. **3** (1955–56), 123–132.

Si $\lambda(t)$ est régulière sur le segment $(-i\pi, i\pi)$, on peut définir l'opérateur $\lambda(D)$ pour les fonctions f de type exponentiel π, à croissance polynomiale sur l'axe réel. L'auteur établit que la condition $\lambda(\pm i\pi)=0$ est nécessaire et suffisante pour que les hypothèses $f(n)=O(1)$ $(-\infty<n<\infty)$ et $f(x)=o(|x|)$ entrainent

$$|\lambda(D)f(x)|<C \sup |f(n)|.$$

D'autres théorèmes sont relatifs au cas $f(x)=o(|x|^q)$, et généralisent à la fois des résultats de Timan [Dokl. Akad. Nauk SSSR (N.S.) **89** (1953), 17–20; MR **14**, 966] et de Macintyre [Proc. London Math. Soc. (2) **45** (1938), 1–20]. *J. P. Kahane* (Montpellier).

Pandey, Nirmala. On two divergent Dirichlet's series. Bull. Calcutta Math. Soc. **47** (1955), 209–210.

The author considers the divergent series

$$\sum \exp\{Ain+k(n/\log n)-sn^\beta\}\ (0<A<2\pi,\ 0<\beta<1),$$

$$\sum \exp\{Ain^\alpha+k(n/\log n)^\beta-sn^\gamma\}\ (A>0,\ 0<\gamma<\beta\leqq\alpha<1).$$

If k is replaced by $k-y$, the series converge for $\Re(y)>\delta$, and the author shows that they represent entire functions $H(s, y)$ of (s, y), so that $\lim_{y\to 0} H(s, y)$ is in each case an entire function of s, which may be regarded as a sum of the original series. *R. P. Boas, Jr.*

Pandey, Nirmala. On the analytic continuation of Dirichlet's series. Bull. Calcutta Math. Soc. **47** (1955), 211–215.

Let $H(s)=\sum_p^\infty \phi(n)e^{-sf(n)}$, where $f(n)>0$, $f(z)$ and $\phi(z)$ are analytic in the angle $\psi_2\leqq\arg(z-h)\leqq\psi_1$, $p-1<h<p$, $0<\psi_1\leqq\pi/2$, $-\pi/2\leqq\psi_2<0$; then if $f(z)\sim z/\log z$ and $\phi(z)=O(|e^{kf(z)}|e^{A|y|})$, $A<2\pi$, we have $H(s)=G(s)+J(s)$, where $G(s)$ is entire and $J(s)=\int_p^\infty \phi(x)e^{-sf(x)}dx$. If $f(z)\sim(z/\log z)^\beta$ and $\phi(z)=O(|e^{Kf(z)}|\exp(-D|y|^\alpha))$, $0<\beta\leqq\alpha\leqq1$, then $H(s)$ is entire. For example,

$$\sum_2^\infty \exp\{Ain+(K-s)n/\log n\}$$

represents an entire function if $0<A<2\pi$.

R. P. Boas, Jr. (Evanston, Ill.).

Schaeffer, A. C. Entire functions. Pacific J. Math. **6** (1956), 351–362.

Let N be a sequence of (positive and negative) integers. The author asks for conditions on N equivalent to the property that for every entire function f of exponential type less than π the requirement $|f(n)|\leqq1$ for $n \in N$ implies that f is bounded on the real axis. Let $\lambda(t)$ be the greatest integer μ such that every real interval of length t contains at least μ elements of N. Then $\lim \lambda(t)/t=1$ is the desired condition. The author then answers a question of the reviewer's by constructing an entire function f of exponential type such that $|f(x)|$ approaches a finite limit as $x\to\infty$ but $|f(x+iy)|$ does not approach a limit for $y\neq0$. This suggests the following theorem which the author proves: if $f(z)$ is an entire function of exponential type and both $|f(x)|$ and $|f^{(m)}(x)|$ approach finite limits as $x\to\infty$ then so does $|f^{(n)}(x+iy)|$ for all n and y. *R. P. Boas, Jr.*

Singh, S. K. Exceptional values of entire functions. Duke Math. J. **23** (1956), 527–531.

Let $L(t)$ be continuous for large t and let $L(ct)\sim L(t)$ for each positive c. The author proves the following theorems. If f is an entire function of finite positive order ρ, and $\lim\sup \{\log M(r)\}/\{r^\rho L(r)\}=a$, then

$$\lim\sup n(r, x)/\{r^\rho L(r)\}\geqq K(\rho)a$$

with an explicit $K(\rho)$, except perhaps for one value of x. If $T(r)$ denotes the Nevanlinna characteristic, then

$$T(r)/\{r^\rho L(r)\}\to1 \quad\text{and}\quad n(r, x_1)/\{r^\rho L(r)\}\to0$$

imply $n(r, x)/\{r^\rho L(r)\}\to\rho$ for all $x\neq x_1$. If $n(r, x)/\{r^\rho L(r)\}\to0$ for two values of x, then the same is true for all x. Some corollaries are given. *R. P. Boas, Jr.* (Evanston, Ill.).

Rahman, Qazi Ibadur. On the maximum modulus and the coefficients of an entire Dirichlet series. Tôhoku Math. J. (2) **8** (1956), 108–113.

Let $f(s)$ be an entire function defined by a Dirichlet series $\sum a_n e^{-\lambda_n s}$, where $\lim\sup \lambda_n^{-1} \log n=D<\infty$. The author establishes numerous relations involving $M(\sigma)=\sup |f(\sigma+it)|$; $\mu(\sigma)$, the maximum term; and $\lambda_{\nu(\sigma)}$, the λ_n corresponding to the maximum term. These are analogous to known results for entire functions defined by power series. For example, if l_k denotes the kth iterate of $\log x$, $k\geqq2$, and L stands for either lim inf or lim sup, then

$$\mathrm{L}\ l_k\mu(\sigma)/l_{k-1}\sigma=\mathrm{L}\ l_kM(\sigma)/l_{k-1}\sigma=1+\mathrm{L}\ l_{k-1}\lambda_{\nu(\sigma)}/l_{k-1}\sigma$$

or

$$\max(1, \mathrm{L}\ l_{k-1}\lambda_{\nu(\sigma)}/l_{k-1}\sigma),$$

according as $k=2$ or $k>2$. *R. P. Boas, Jr.*

Šilov, G. E. On a boundary property of analytic functions. Moskov. Gos. Univ. Uč. Zap. 145, Mat. 3 (1949), 126–128. (Russian)

Let $u(z)$ be an analytic function in $|z|<1$ which is continuous in $|z|\leq 1$, vanishes at the single point $z=1$, and has zero logarithmic residue at that point. Then, it was shown by T. Carleman [Les fonctions quasi-analytiques, Gauthier-Villars, Paris, 1926; Note III, 107–109] that, in the terminology of the theory of rings, in the ring A of all functions, analytic in $|z|<1$, continuous in $|z|\leq 1$, the principal ideal generated by $u(z)$ contains any function $v(z)\in A$ which vanishes at $z=1$. The author considers the analogous question if $u(z)$ has a non-vanishing logarithmic residue and proves the following theorem: The functions

$$u_\alpha(z)=(z-1)\exp\left(\frac{\alpha}{z-1}\right)(0\leq\alpha<\infty)$$

belong to the ring A and vanish at the single point $z=1$. They generate distinct closed ideals in the ring A. Every function $u(z)\in A$ which vanishes at the single point $z=1$ generates an ideal which coincides with one of the above ideals. *W. Seidel* (Notre Dame, Ind.).

Erdös, P.; and Rényi, A. On the number of zeros of successive derivatives of analytic functions. Acta Math. Acad. Sci. Hungar. 7 (1956), 125–144. (Russian summary)

Let $N_k(r)$ denote the number of zeros of $f^{(k)}(z)$ in $|z|\leq r$. The authors prove several theorems on the asymptotic behavior of $N_k(r)$, generalizing previous results of Pólya [Bull. Amer. Math. Soc. 49 (1943), 178–191; MR 4, 192] and Evgrafov [Interpolyacionnaya zadača Abelya-Gončarova, Gostehizdat, Moscow, 1954; MR 16, 1104]. They also discuss r_k, the modulus of the zero of $f^{(k)}(z)$ that is closest to the origin. Theorem 1. If $f(z)$ is regular in $|z|<1$, then $\liminf k^{-1}N_k(r)\leq K(r)$, where $K=K(r)$ is the positive root of $r=K(1+K)^{-1-1/K}$. More precisely, this is true when k runs through the values for which $f^{(k)}(0)\neq 0$, and in this form it is best possible. For an entire function, Theorem 1 gives $\liminf k^{-1}N_k(1)=0$. If the growth of the entire function is restricted, more can be said. Theorem 2. Let $g(r)\uparrow\infty$ and let h be its inverse function. If

$$\liminf\{\log M(r)\}/g(r)<1,$$

then $\liminf k^{-1}h(k)N_k(1)\leq e^2$. This is essentially best possible. Let f be any entire function and let H be the inverse of $\log M(r)$. Then $\liminf k^{-1}H(k)N_k(1)\leq e^2$ and (Theorem 3) $\liminf k^{-1}H(k)/r_k\leq e/\log 2$. Theorem 3 generalizes a theorem of Ålander's [Uppsala thesis, 1914]. Theorem 4. If f is regular in $|z|<R$ and not a polynomial, then $\limsup kr_k\geq R\log 2$. The authors remark that $\sum r_k$ always diverges (except for polynomials) and hence [conjectured by Erwe, Arch. Math. 7 (1956), 55–58; MR 17, 835] if $f^{(n)}(z_n)=0$ with $|z_{n+1}|\leq\frac{1}{2}|z_n|$ then f is a polynomial. The authors conjecture that for entire functions $\limsup(r_1+\cdots+r_k)/\log k=\infty$. *R. P. Boas, Jr.*

Rényi, Catherine. On a conjecture of G. Pólya. Acta Math. Acad. Sci. Hungar. 7 (1956), 145–150. (Russian summary)

Let $f(z)$ be a transcendental entire function. The author establishes Pólya's conjecture that the power series expansions of f around two different points cannot both have Fabry gaps. The proof depends on a theorem of Pólya's, whose first published proof is in the paper reviewed above. More generally, let $Z_a(n)$ denote the number of vanishing coefficients in the first $n+1$ coefficients of the power series of f around $z=a$. Then if $a\neq b$, $\liminf n^{-1}\{Z_a(n)+Z_b(n)\}\leq 1$. *R. P. Boas, Jr.*

Levin, B. Ya. Distribution of roots of exponential sums. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 20–22. (Russian)

Let J denote a bounded convex domain in the complex plane and let $K(\theta)$ be its supporting function. The author considers the class of entire functions $f(z)$ for which the norm $\|f(z)\|_J=\sup|f(re^{i\theta})|e^{-K(\theta)r}$ $(r\geq 0,\ 0\leq\theta<2\pi)$ is finite, and which can be represented as a limit in the sense of this norm of a sequence of exponential sums $P_n(z)=\sum_1^{m_n}a_n(\lambda_k)e^{\lambda_k z}$, where the exponents are complex and their complex conjugates are situated on the boundary of J. Every function $f(z)$ has a spectrum on the boundary of J, and if J is the convex hull of the spectrum, the following statement holds: If $n(r,\vartheta,\theta)$ is the number of zeros of $f(z)$ in the sector $\vartheta<\arg z\leq\theta$, $|z|\leq r$, then $\Delta(\vartheta,\theta)=\lim_{r\to\infty}r^{-1}n(r,\vartheta,\theta)$ exists when the half-lines $\arg z=\vartheta$ and $\arg z=\theta$ are not perpendicular on a straight segment on the boundary of J. The limit $\Delta(\vartheta,\theta)$ is equal to $(2\pi)^{-1}$ times the length of the arc of the boundary of J between the contact points with the supporting lines perpendicular on $\arg z=\vartheta$ and $\arg z=\theta$. The conditions are in particular satisfied for a function $f(z)=\sum_1^\infty a_ke^{\lambda_k z}$, where $\sum|a_k|$ converges, and for a function $f(z)=\sum_1^\infty(\varphi(n)/n!)z^n$, where $\varphi(x)$ is an almost periodic function. In the last case the domain J is the unit circle. No proofs are given. *H. Tornehave* (Virum).

Collingwood, E. F. A theorem on prime ends. J. London Math. Soc. 31 (1956), 344–349.

Let D be a simply connected domain. Suppose that $f(z)$ maps the disc $|z|<1$ conformally onto D. Denote by $C(f,e^{i\theta})$ the cluster set of $f(z)$ at $e^{i\theta}$, as usual, and by $C_\rho(f,e^{i\theta})$ the radial cluster set of $f(z)$ at $e^{i\theta}$. The prime end E of D corresponding to a point $z=e^{i\theta}$ is $C(f,e^{i\theta})$. A classical theorem of Lindelöf shows that $C_\rho(f,e^{i\theta})$ is the set of principal points of the prime end $E=C(f,e^{i\theta})$. A prime end E is of the first kind if $C(f,e^{i\theta})=C_\rho(f,e^{i\theta})$ and $C_\rho(f,e^{i\theta})$ is degenerate; of the second kind if $C(f,e^{i\theta})\neq C_\rho(f,e^{i\theta})$ and $C_\rho(f,e^{i\theta})$ is degenerate; of the third kind if $C(f,e^{i\theta})=C_\rho(f,e^{i\theta})$ and $C_\rho(f,e^{i\theta})$ is non-degenerate; of the fourth kind if $C(f,e^{i\theta})\neq C_\rho(f,e^{i\theta})$ and $C_\rho(f,e^{i\theta})$ is non-degenerate. Let $\mathcal{E}$ denote the set of prime ends, and $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ and $\mathcal{E}_4$ denote the subsets of the prime ends of these four kinds respectively. Adopt as the metric in $\mathcal{E}$ the distance between the pair of points on $|z|=1$ corresponding to a pair of prime ends in $\mathcal{E}$. Then $\mathcal{E}$ is a complete metric space and so of category II. The main purpose of this paper is to prove the following: For any simply connected domain, the union $\mathcal{E}_1\cup\mathcal{E}_3$ is a residual subset of the set $\mathcal{E}$ of all the prime ends of D and corresponds under any conformal mapping of $|z|<1$ onto D to a residual set on $|z|=1$. This is an immediate consequence of the author's beautiful theorem: $C(f,e^{i\theta})=C_\rho(f,e^{i\theta})$ for all values of θ belonging to a residual subset of the interval $0\leq\theta<2\pi$ [C.R. Acad. Sci. Paris 240 (1955), 1502–1504; MR 16, 916]. This theorem also contains an answer to Carathéodory's question [Math. Ann. 73 (1913), 323–370, p. 325] as to whether there exists a domain D such that $\mathcal{E}=\mathcal{E}_2$. *K. Noshiro* (Nagoya).

Waadeland, Haakon. Bemerkung zu einer Arbeit von Golusin. Norske Vid. Selsk. Forh., Trondheim 29 (1956), 29–32.

G. M. Golusin has shown [Mat. Sb. N.S. 3(45) (1938),

321–330] that if $f(z)=z+a_{n+1}z^{n+1}+\cdots$ is schlicht and regular in $|z|<1$, then $|a_{p+1}|\leq 2/p$, $n\leq p\leq 2n-1$. By means of easy transformations, A. Rosenblatt then showed [Actas Acad. Ci. Lima 4 (1941), 145–155; MR 4, 7] that as a consequence if the odd function $f(z)=z+a_{2n+1}z^{2n+1}+a_{2n+3}z^{2n+3}+\cdots$ is schlicht and regular in $|z|<1$ then $|a_{2p+1}|\leq 1/p$, $n\leq p\leq 2n-1$. The author now points out that the substitutions $2n=m$, $2p=q$ in the former result show the latter result actually to be included in the former. Finally, he points out that the following result is equivalent to that of Golusin: If $f(z)=z+a_{nk+1}z^{nk+1}+\cdots$ is schlicht and regular in $|z|<1$, then $|a_{pk+1}|\leq 2/(pk)$, $n\leq p\leq 2n-1$. Extremal functions are given in all instances.

E. F. Beckenbach (Los Angeles, Calif.).

Bowen, N. A. Some limit theorems for bounded analytic functions. J. London Math. Soc. 31 (1956), 437–445.

Let $F(z)$ be analytic and satisfy $|F(z)|\leq 1$ in $y>0$. Then Montel's theorem and its refinements state that if $F(z)\to 0$ as $z\to\infty$ along a sufficiently dense set, $F(z)\to 0$ uniformly in any interior angle. The object of this paper is to obtain alternative conditions when the given set is a set of zeros of F, conditions which turn out also to be necessary for the subclass consisting of Blaschke products. Let the zeros of F ($\not\equiv 0$) be $a_\mu=x_\mu+iy_\mu$, $|a_\mu|\uparrow\infty$; then $\sum y_\mu|a_\mu|^{-2}<\infty$. Let $\phi(z)=\prod \bar{a}_\mu(z-a_\mu)/\{a_\mu(z-\bar{a}_\mu)\}$, so that $|F(z)|\leq|\phi(z)|$ if $F(z)$ is any other function satisfying the given hypotheses with the same zeros. Then $\phi(z)\to 0$ as $z\to\infty$ along a curve Γ if and only if

$$S(z)=\sum yy_\mu/\{y^2+|z-a_\mu|^2\}\to+\infty$$

as $z\to\infty$ along Γ. Let $n(t)$ be the number of a_μ with $|a_\mu|<t$, $N(t)$ the number of a_μ with $x_\mu\leq t$. Under further hypotheses, $S(z)$ can be replaced by

$$T(z)=\int_1^\infty v^{-2}\{n(|z|v-n(|z|/v)\}dv$$

or by

$$U(z)=\int_1^\infty v^{-3}\{N(|z|+v)-N(|z|-v)\}dv.$$

Finally the author shows that for any analytic bounded function in $y>0$, there is at most one value of a for which the a-points in an interior angle satisfy $T(z,a)\to\infty$ or the a-points in an interior half-strip $x>0$, $0<\delta<y<1/\delta$, satisfy $U(z,a)\to\infty$. In particular, the a-points in $y\geq 1$ satisfy $t^{-1}n(t,a)\to\infty$ for at most one a. *R. P. Boas, Jr.*

Eremin, S. A. On the construction of complete systems of functions of two complex variables. Ukrain. Mat. Ž. 8 (1956), 214–217. (Russian)

A. Temlyakov [Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. 20 (1954), 7–16; MR 17, 838] established a one-to-one correspondence between the set of functions analytic in the bicylinder $|x|<R$, $|y|<R$ and the set of functions analytic in the hypercone $|x|+|y|<R$. The author of the present paper proves that this one-to-one correspondence maps a set of functions complete in the bicylinder with respect to uniform convergence on every closed subset onto a set of functions with the same property in the hypercone. *H. Tornehave* (Virum).

See also: R.-Salinas, p. 204; Backus, p. 206; Wuyts, p. 209; Kapanyan, p. 247; Kovácsand Solymár, p. 257.

Functions with Particular Properties

Mergelyan, S. N. Harmonic approximation and approximate solution of Cauchy's problem for Laplace's equation. Dokl. Akad. Nauk SSSR (N.S.) 107 (1956), 644–647. (Russian)

Let σ be a smooth surface lying in an open sphere G in 3-space, and let $\gamma(\delta)$ be the modulus of smoothness of σ, i.e. the supremum of the magnitude of the angle between the normals to σ at P_1 and P_2 for $|P_1P_2|\leq\delta$. Further, let f_1 and f_2 be continuous real-valued functions on σ, and let $\omega(\delta)$ be their joint modulus of continuity:

$$\omega(\delta)=\max_{i=1,2}\sup|f_i(P_1)-f_i(P_2)| \text{ for } |P_1P_2|\leq\delta.$$

The infimum of the expression

$$\sup_{P\in\sigma}|f_1(P)-H(P)|+\sup_{P\in\sigma}\left|f_2(P)-\frac{\partial H}{\partial n}\right|$$

1) taken over all harmonic polynomials H of degree not exceeding m will be denoted by $E_m(f_i,\sigma)$; 2) taken over all harmonic functions H on G such that $|H|\leq M$ will be denoted by $\mathcal{E}_M(f_i,\sigma)$.

A number of approximation results based on estimates for $E_m(f_i,\sigma)$ and $\mathcal{E}_M(f_i,\sigma)$ are given. The following two theorems are typical. Theorem: For any $\varepsilon>0$ there exist constants C_1 and C_2 such that

$$E_m(f_i,\sigma)<C_1\omega[\gamma^{1-\varepsilon}(1/m^{1-\varepsilon})],$$
$$\mathcal{E}_M(f_i,\sigma)<C_2\omega[\gamma^{1-\varepsilon}(1/(\log M)^{1-\varepsilon})].$$

Theorem: Let σ be homeomorphic to the surface of a sphere. Then for

$$\lim_{m\to\infty}E_m(f_i,\sigma)=0$$
$$\lim_{M\to\infty}\mathcal{E}_M(f_i,\sigma)=0$$

it is necessary and sufficient that

$$\iint_\sigma [f_1(Q)\frac{\partial}{\partial n}(1/r_{PQ})-f_2(Q)(1/r_{PQ})]d\sigma=0$$

for all points P in the unbounded region determined by σ.
M. G. Arsove (Seattle, Wash.).

Besov, O. V. On some properties of harmonic functions given in a half-space. Izv. Akad. Nauk SSSR. Ser. Mat. 20 (1956), 469–484. (Russian)

Relationships between harmonic functions U on the half-space $x_n>0$ in R^n and the corresponding boundary functions φ are derived. In each case φ is assumed to belong to an appropriate Nikol'skiĭ class and information is obtained on the L^p behavior of U and its generalized derivatives. For example (Theorem 2), if φ is in $H_p^{(r+\varepsilon-1/p)}$ for p, $r\geq 1$ and $0<\varepsilon<1/p$, then all generalized derivatives of U of orders 1 through r have finite L^p norm on the half-space. *M. G. Arsove* (Seattle, Wash.).

McMahon, James. Lower bounds for the Dirichlet integral in Euclidean n-space. Proc. Roy. Irish Acad. Sect. A. 58 (1956), 1–12.

The concept of pyramid vectors introduced by J. L. Synge [same Proc. 54 (1952), 341–367; MR 14, 801] for two-dimensional problems and previously generalized to three-dimensions by the author [ibid. 55 (1953), 133–167; MR 15, 425] is here generalized to n-dimensions. It is shown that an arbitrarily close lower bound for n-dimensional capacity is obtained by using linear combi-

nations of the gradient of r^{-n+2} and pyramid vectors in Thomson's principle.

H. Weinberger.

Matsushita, Shin-ichi. Fonctions presque périodiques du type spécial. I, II, III, IV, V, VI. Proc. Japan Acad. 31 (1955), 70–75, 156–160, 214–219, 278–283, 334–339, 436–440.

Let G be a locally compact group and $L_1(G)$ taken with respect to, say, left Haar measure. The space V of all continuous positive definite functions ϕ on G such that $\phi(e)=1$ may be thought of as part of the conjugate space $L_1^*(G)$. Let V_0 be the weak closure of V in $L_1^*(G)$, except for the function zero. Then V_0 consists of continuous positive definite functions ϕ such that $\phi(e)=1$ (and, as Yoshizawa has shown [Osaka Math. J. 3 (1951), 55–63; MR 13, 10], V_0 may consist of all continuous positive definite functions ϕ such that $\phi(e)=1$). With the topology of uniform convergence on compact subsets of G, V_0 is a locally compact space. Let M be the linear space of all Radon measure on V_0, topologized with the vague topology (i.e., the weak topology using continuous functions on V_0 that vanish outside of compact sets). Let $\mathfrak{M}^1$ be the space of all bounded Radon measures on V_0, topologized with the weak topology based on continuous functions on V_0 that vanish at infinity.

The aim of the notes under review is to study continuous bounded mappings of G into $\mathfrak{M}$ ($\mathfrak{M}^1$) that are almost periodic in the sense of Bochner and von Neumann [Trans. Amer. Math. Soc. 37 (1935), 21–50]. Let $\mathfrak{A}(G)$ ($\mathfrak{A}^1(G)$) be space of all bounded mappings of G into $\mathfrak{M}$ ($\mathfrak{M}^1$) that are almost periodic. Since $\mathfrak{M}$ and $\mathfrak{M}^1$ need not be metrizable, the theory of $\mathfrak{A}(G)$ and $\mathfrak{A}^1(G)$ cannot be immediately subsumed under the Bochner-von Neumann theory. Theorem 1. Let $x\to\alpha(x)$ be a bounded continuous mapping of G into $\mathfrak{M}$ ($\mathfrak{M}^1$). Then α is in $\mathfrak{A}(G)$ ($\mathfrak{A}^1(G)$) if and only if the complex-valued function

$$\langle f, \alpha(x)\rangle = \int_{V_0} f(\phi)d\alpha(x)(\phi)$$

is almost periodic for all continuous functions on V_0 vanishing outside of compact sets. The proof, which cites a theorem of Bochner and von Neumann [loc. cit., Th. 5], would appear to need closer examination in view of the possible non-fulfilment of one of the axioms of Bochner and von Neumann. A theorem of the reviewer [Proc. Amer. Math. Soc. 4 (1953), 663–670; MR 15, 119] and a theorem of L. Schwartz [Théorie des distributions, t. II, Hermann, Paris, 1951, p. 62; MR 12, 833] appear as corollaries.

The Bochner-von Neumann theory (loc. cit.) is next reproduced nearly in toto for $\mathfrak{A}(G)$ and $\mathfrak{A}^1(G)$. The existence and uniqueness of invariant means, expansions in Fourier series, uniqueness of Fourier coefficients, Parseval's relation, classification of closed invariant subspaces, are all established. Of the Bochner-von Neumann theory, only the summability theorem is missing.

The author uses his methods to reproduce a number of other known results; for example, Tanaka's duality theorem, and Eberlein's weakly almost periodic functions [Trans. Amer. Math. Soc. 67 (1949), 217–240; MR 12, 112] are treated. In addition, the author gives a large number of new definitions and theorems concerning the objects defined. For the details, we refer the reader to the original notes.

E. Hewitt (Seattle, Wash.).

Kovan'ko, A. S. On compactness of systems of the almost periodic functions of B. M. Levitan. L'vov. Gos. Univ. Uč. Zap. 29, Ser. Meh.-Mat. no. 6 (1954), 45–49. (Russian)

Let S be a set of functions on $]-\infty, \infty[$ that are N-almost periodic, in the sense of B. M. Levitan [Almost periodic functions, Gostehizdat, Moscow, 1953; MR 15, 700]. S is said to be equi N-almost periodic if for every $\varepsilon>0$ and $N>0$, the following hold: 1) there is a number $M(N)>0$ such that $|f(x)|<M$ for all $f\in S$ and $|x|<M$; 2) there exists a $\delta(\varepsilon, N)$ such that $|f(x+h)-f(x)|<\varepsilon$ for all $f\in S$, $|h|<\delta$, and $|x|<N$; there exists a relatively dense set of almost periods $\tau=\tau(\varepsilon, N)$ for all $f\in S$. The principal result of the note is that every equi N-almost periodic set is relatively compact under the topology of uniform convergence on bounded sets in $]-\infty, \infty[$. *E. Hewitt.*

Kovan'ko, A. S. On expandability of almost periodic functions in a finite sum of almost periodic functions. L'vov. Gos. Univ. Uč. Zap. 22, Ser. Fiz.-Mat. no. 5 (1953), 12–16. (Russian)

Let f be a continuous almost periodic function on $]-\infty, \infty[$ with Fourier series $\sum_{k=1}^{\infty} A_k e^{i\lambda_k x}$. Suppose that the set of real numbers $\{\lambda_k\}_{k=1}^{\infty}$ can be written in the form $X_1\cup X_2$, where $X_1\cap X_2=0$ and X_1 and X_2 have disjoint bases over the integers. Then the series $\sum_{\lambda_k\in X_j} A_k e^{i\lambda_k x}$ are the Fourier series of continuous almost periodic functions f_j ($j=1, 2$), and $f=f_1+f_2$. *E. Hewitt.*

Weston, J. D. Almost periodic functions. Mathematika 2 (1955), 128–131.

Let T be a topological group and M a complete metric space with distance function $|x, y|$. Let $B(T)$ be the set of all bounded mappings of T into M, with distance function $\|f, g\| = \sup\{|f(t), g(t)| : t\in T\}$. For $f\in B(T)$ and $t\in T$, let f^t be the element of $B(T\times T)$ such that $f^t(u, v)=f(utv)$, for all $(u, v)\in T\times T$. Let $S_f=\{f^t : t\in T\}$. A continuous function $f\in B(T)$ is said to be almost periodic if $\bar{S}_f$ is compact in the metric space $B(T\times T)$. Let $A(T)$ be the set of all almost periodic functions. The first theorem asserts that the weakest topology for T in which the group operations are continuous and all functions in $A(T)$ are continuous is the topology in which $\{t : t\in T, \|f^t, f\|<\varepsilon\}$ ($f\in A(T)$, $\varepsilon>0$) are an open sub-basis for neighborhoods of e. This amounts, of course, to the assertion that the group operations are continuous in this topology.

Suppose now that the topology just described satisfies Hausdorff's separation axiom, that is, that for $s\neq t$, there is an $f\in A(T)$ such that $f(s)\neq f(t)$. Then T with the weak topology can be imbedded as a dense subgroup of a compact group $T^{\#}$ in such a way that every function in $A(T)$ admits a continuous extension over $T^{\#}$ and every continuous M-valued function on $T^{\#}$ is the extension of some function in $A(T)$. Both results of the present note are of course well-known for complex-valued almost periodic functions. *E. Hewitt* (Seattle, Wash.).

Faleschini, Bruno. Sulle definizioni e proprietà delle funzioni a variazione limitata di due variabili. I, II. Boll. Un. Mat. Ital. (3) 11 (1956), 80–92, 260–275.

The author lists the many definitions of monotony and bounded variation for functions of two variables and gives some 90 references. *L. C. Young* (Madison, Wis.).

See also: Reade, p. 199; Kusunoki, p. 200; Šilov, p. 201; Suharevskiĭ, p. 205; Cremonesi, p. 206; Backes, p. 230; Hájek, p. 241; Nowinski, p. 249; Mackie, p. 251.

Special Functions

★**Sneddon, Ian N. Special functions of mathematical physics and chemistry.** Oliver and Boyd, Edinburgh and London; Interscience Publishers, Inc., New York, 1956. viii+164 pp.

This book is intended primarily for the student of applied mathematics, physics, chemistry or engineering who wishes to use the 'special' functions associated with the names of Legendre, Bessel, Hermite and Laguerre. It aims at providing in a compact form most of the properties of these functions which arise most frequently in applications, and at establishing these properties in the simplest possible way. For that reason the methods it employs should be intelligible to anyone who has completed a first course in calculus and has a slight acquaintance with the theory of differential equations. Use is made of the theory of functions of a complex variable only very sparingly, and most of the book should be accessible to a reader who has no knowledge of this theory. Throughout the text an attempt is made to show how these functions may be used in the discussion of problems in classical physics and in quantum theory. A brief account is given in an appendix of the main properties of the Dirac delta 'function'.

From the preface.

Brödel, Walter. Zur Theorie der Laméschen Funktionen. Wiss. Z. Friedrich-Schiller-Univ. Jena **5** (1955/56), 151–155.

The author constructs Lamé polynomials by constructing, in confocal coordinates, normal solutions of Laplace's equation regular everywhere, and using the connection between spherical and ellipsoidal harmonics. [Cf. A. Erdélyi et al., Higher transcendental functions, v. 3, McGraw-Hill, New York, 1955, Ch. XV and the literature quoted there; MR **16**, 586.] *A. Erdélyi* (Jerusalem).

Singh, V. N. A note on the partial sums of certain basic bilateral hypergeometric series. Proc. Cambridge Philos. Soc. **52** (1956), 756–758.

Agarwal [same Proc. **49** (1953), 441–445; MR **15**, 122] gave a transformation of a particular truncated basic hypergeometric series, ${}_8\Phi_7$, into a constant multiple of another truncated ${}_8\Phi_7$. The author shows how this result may be used to derive similar results for truncated bilateral hypergeometric series. *N. D. Kazarinoff.*

Saran, Shanti. The solutions of certain hypergeometric equations. Proc. Nat. Inst. Sci. India. Part A. **21** (1955), 404–408 (1956).

The author obtains expansions for the general solution of the differential equations of two of the ten hypergeometric functions of three variables, which he has previously defined [Acta Math. **93** (1955), 293–312; Proc. Nat. Inst. Sci. India. Part A. **21** (1955), 83–90; MR **17**, 150, 262]; he uses some Pochammer-integral representations to investigate these solutions.

J. Kampé de Fériet (Lille).

See also: Newman, p. 194; Cohn, p. 194; Whiteman, p. 195; Gundlach, p. 195; Dickinson; Pollak; and Wannier, p. 206; Rathie, p. 208; Leray, p. 215; Darling, p. 240; Nowacki and Olesiak, p. 248; Piątek, p. 250; Schmieden und Müller, p. 252; Jung, p. 264.

Sequences, Series, Summability

R.-Salinas, Baltasar. Functions with null moments. Rev. Acad. Ci. Madrid **49** (1955), 331–368. (Spanish)

The author discusses questions of determinacy for several modifications of the Stieltjes moment problem. (i) Let $F(z)$ be the restriction to the real axis of a function analytic and bounded for $|\arg z|<\pi\alpha/2$. If

$$\text{(a)} \qquad \int_0^\infty |F(x)|x^n dx \leqq m_n \ (n=0, 1, \cdots)$$

and if (b) $\int_0^\infty F(x)x^n dx=0$, then $\liminf_{n\to\infty} (m_n)^{1/n} n^{-\alpha-2}=0$ implies that $F(x)\equiv 0$. Conversely, if

$$\liminf_{n\to\infty} (m_n)^{1/n} n^{-\alpha-2}>0,$$

then there exists $F(z)\not\equiv 0$ analytic and bounded for $|\arg z|<\pi\alpha/2$ and satisfying (a) and (b). (ii) Let $F(t)$ be summable on the two rays $t=re^{i\phi_1}$, $t=re^{i\phi_2}$ ($\phi_2-\phi_1=\alpha_0\pi$, $0<\alpha_0\leqq 1$) and let (c) $\int_0^{\infty(\phi_1)} F(t)t^n dt=\int_0^{\infty(\phi_2)} F(t)t^n dt$, and (d) $\int_0^{\infty(\phi_k)} |F(t)t^n dt|\leqq m_n$ ($k=1, 2$; $n=0, 1, \cdots$). If $\int_1^\infty \log T(r) r^{-1-\alpha_0^{-1}} dr=+\infty$, where $T(r)=\sup_n r^n/m_n$, then $F(t)\equiv 0$. On the other hand if this integral is convergent then there exists $F(t)\not\equiv 0$, satisfying (c) and (d). (iii) Let $f(z)$ be analytic for $|\arg z|<\alpha\pi/2$, let (e) $f^{(n)}=0$ ($n=0, 1, \cdots$) hold for the angular derivatives in $|\arg z|<\alpha\pi/2$, and let (f) $|f^{(n)}(z)|\leqq m_n$ ($n=0, 1, \cdots$). If $\log T(r) r^{-1-(\alpha+1)^{-1}} dr=+\infty$, then $f(z)\equiv 0$. Conversely if this integral is finite then there exists a function $f(z)\not\equiv 0$ analytic in $|\arg z|<\alpha\pi/2$ and satisfying (e) and (f). Further applications and variations of these ideas are given. *I. I. Hirschman.*

Wright, Fred M. On the backward extension of positive definite Hamburger moment sequences. Proc. Amer. Math. Soc. **7** (1956), 413–422.

The statement that $\{c_n\}_{n=0}^\infty$ is an H-sequence means there exists a function ϕ, nondecreasing on the set of all numbers and having infinitely many points of increase, such that $c_n=\int_{-\infty}^{+\infty} u^n d\phi(u)$ ($n=0, 1, 2, \cdots$). H. Hamburger [Math. Ann. **81** (1920), 235–319; **82** (1920), 120–164, 168–187 (1921)] developed a general theory of these moment sequences following the pattern of Stieltjes. E. Hellinger [Math. Ann. **86** (1922), 18–29] developed a general theory based on Hilbert's theory of infinite linear systems. A first backward extension of the H-sequence $\{c_n\}_{n=0}^\infty$ is an H-sequence $\{d_n\}_{n=0}^\infty$ such that $c_n=d_{n+2}$ ($n=0, 1, 2, \cdots$). Necessary and sufficient conditions for an H-sequence to possess backward extensions were obtained by the reviewer [Trans. Amer. Math. Soc. **33** (1931), 511–532] by means of Hamburger's theory. The author obtains new necessary and sufficient conditions by means of Hellinger's "Theorem of invariability". He uses his results to obtain a new proof of Hamburger's criterion for the moment problem to be determinate.

H. S. Wall (Austin, Tex.).

Keller, Joseph B.; Kay, Irvin; and Shmoys, Jerry. Determination of the potential from scattering data. Phys. Rev. (2) **102** (1956), 557–559.

It is shown that in classical mechanics a spherically symmetric repulsive potential is determined from the differential scattering cross section for particles of a single energy E. (The potential is not determined within the sphere of radius of closest approach at energy E.) The problem is to solve

$$\theta(b)=\pi-2\int_0^\infty r^{-2}[b^{-2}-r^{-2}-V(r)E^{-1}b^{-2}]^{-\frac{1}{2}}dr$$

for $V(r)$ when $\theta(b)$ is given. This is done by transforming it to an integral equation of Abel type that can be solved explicitly. *N. Levinson* (Cambridge, Mass.).

Mysovskih, I. P. Proof of the existence of an eigenvalue for a symmetric kernel. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 2(68), 199–200. (Russian)

The author derives the theorem described in the title of the paper from the fact that the Neumann series for the resolvent kernel of the integral equation

$$\varphi(s)=f(s)+\lambda\int_a^b K(s,t)\varphi(t)\,dt$$

is convergent for all λ such that $|\lambda|<|\lambda_1|$, where λ_1 is the absolutely smallest eigenvalue. *F. Smithies.*

Barsotti, Leo. Series expansion of the arc sine function. Soc. Parana. Mat. Anuário 2 (1955), 1–2. (Portuguese)

The author deduces the Maclaurin series for $\sin^{-1} x$ from that of $\tan^{-1} x$ by manipulating series, in order to obtain the first series without applying calculus to power series. *R. P. Boas, Jr.* (Evanston, Ill.).

Wintner, Aurel. On an absolute constant pertaining to Cauchy's "principal moduli" in bounded power series. Math. Scand. 4 (1956), 108–112.

Let $f(z)$ be a function which is regular within the unit circle and satisfies the inequality $|f(z)|<1$ for $|z|<1$. For $|z|<1$, let another regular function, $g(z)$, be defined by placing $g(z)=\sum_0^\infty |c_n|z^n$ if $f(z)=\sum_0^\infty c_nz^n$. It is clear that $\sup_{|z|<1} |z/f(z)|\geqq 1$ (the sup can be ∞) and that, in view of the example $f(z)=z$, the lower bound 1 cannot be improved to any greater absolute constant. In the present paper it is shown that $\sup_{|z|<1} |z/g(z)|>\frac{1}{2}$ and that $\frac{1}{2}$ is the best absolute constant. *T. Fort* (Columbia, S.C.).

Leont'ev, A. F. On convergence of a sequence of Dirichlet polynomials. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 23–26. (Russian)

Soit $\{\lambda_n\}$ une suite positive telle que $\lim\sup n^{-1}\lambda_n=\sigma$, et $P_n(z)$ une suite de polynômes de Dirichlet, dont les exposants appartiennent à $\{\lambda_n\}$, et qui converge uniformément dans un cercle $|z-z_0|<R$, $R>\pi\sigma$. Alors $P_n(z)$ converge uniformément dans un domaine G, precisé par l'auteur; on peut prendre pour G le domaine $|y|<\exp(\alpha x)$, $x>x_0$ assez grand, avec $\alpha\sigma<1$; mais on ne le peut pas si $\alpha\sigma>1$. L'auteur cherche ensuite des conditions supplementaires, portant sur $\{\lambda_n\}$, permettant de prendre pour G un demi-plan $x>\alpha$. {Remarque: le référent [Ann. Inst. Fourier, Grenoble 5 (1953–1954), 39–130; voir pp. 92–97; MR 17, 732] a établi des théorèmes analogues à ceux de l'auteur, quoique distincts; en particulier, il suffit que $\lambda_{n+1}-\lambda_n>h>0$ pour qu'on puisse prendre G: $\{x>\alpha\}$.} *J. P. Kahane* (Montpellier).

de Castro Brzezicki, A. Some boundary problems for a linear integro-differential equation. Rev. Mat. Hisp.-Amer. (4) 16 (1956), 89–97. (Spanish)

This paper is concerned with the application of Fourier series and transforms to the study of various specific boundary problems involving the linear integro-differential equation

$$y''+ay+\int_{-\infty}^{\infty} f(\tau)y(t-\tau)d\tau=\phi(t),$$

where $f(\tau)\equiv f(-\tau)$. *W. T. Reid* (Evanston, Ill.).

Debi, Sobha. Some results on total inclusion for Nörlund summability. Bull. Calcutta Math. Soc. 47 (1955), 135–141.

Let $(N, p_n>0)$, $(N, q_n>0)$ be regular Nörlund methods. Necessary and sufficient conditions in order that (N, q_n) include (N, p_n) are given by Hardy [Divergent series, Oxford, 1949, p. 67; MR 11, 25]. The author considers the problem of total inclusion, namely, the case in which the generalized limit is allowed to be infinite. He shows that total inclusion is characterized by the conditions stated by Hardy, augmented by the conditions $k_n>0$, where k_n is defined by $\sum k_nx^n=\sum q_nx^n/\sum p_nx^n$. The paper contains other results, bearing on both general methods and certain special methods. It is shown, for instance, that if (N, p_n) and (N, q_n) are totally equivalent, then they are identical. *J. D. Hill* (East Lansing, Mich.).

Mishoe, L. I.; and Ford, G. C. On the uniform convergence of a certain eigenfunction series. Pacific J. Math. 6 (1956), 271–278.

Continuing the investigation begun by Friedman and Mishoe [same J. 6 (1956), 249–270; MR 18, 129] the authors prove that the expansion of a function $F(x)$ in terms of the eigenfunctions of the equation

$$u''+q(x)u+\lambda(p(x)u-u')=0,$$

where $u(0)=u(1)=0$, converges uniformly for $0\leqq x\leqq 1$ if $F'(x)$ is of bounded variation in $0\leqq x\leqq 1$ and if $F(0)=F(1)=0$. *B. Friedman* (New York, N.Y.).

Alexiewicz, A.; and Orlicz, W. On summability of double sequences. I. Ann. Polon. Math. 2 (1955), 170–181 (1956).

The reviewer [Bull. Amer. Math. Soc. **46** (1940), 327–331; MR 1, 219] extended a consistency theorem of Banach [Théorie des opérations linéaires, Warsaw, 1932, p. 95] from single sequence summability to a certain case of double sequence summability. The authors improve this result by removing the hypothesis of reversibility. Other closely related results are given. Notational complexities prevent the giving of relevant details within the scope of a short review. *J. D. Hill.*

Suharevskiĭ, I. V. On convergence of a limiting process in potential theory. Mat. Sb. N.S. **38(80)** (1956), 167–182. (Russian)

Let C be a simple closed contour in the complex plane, smooth except for a possible angular point at z_0; let C' denote a portion of C resulting from the removal of an interval containing z_0. The author considers an integral equation arising from the interior Neumann problem for C, namely

$$(*)\qquad u(z)-\pi^{-1}\int_C u(\zeta)(\partial/\partial n)\log|\zeta-z|d\zeta=f(z)\ (z\in C),$$

and the modified integral equation (**) in which C' replaces C. He first assumes C to be convex, in the wide sense, and shows that the (unique) solution of (**) tends as $C'\to C$ to that solution of (*) for which $u(z_0)=0$; his hypotheses restrict the manner in which $C'\to C$, and require the inclination of the tangent to C to satisfy a Lipschitz condition. A similar result is proved for the case in which C is smooth, but has non-zero curvature at z_0. Finally he shows that (**) is uniquely soluble even if C is not convex, this being deduced from the corresponding potential problem. He appears to have used one of the results in a previous paper [Dopovidi Akad. Nauk Ukrain. SSR **1955**, 39–42; MR 17, 1019]. *F. V. Atkinson* (Canberra).

Bojanić, R.; Jurkat, W.; und Peyerimhoff, A. Über einen Taubersatz für Faltungen. Math. Z. 65 (1956), 195–199.

The reviewer proved [J. Indian Math. Soc. (N.S.) 13 (1949), 131–144, 145–147; MR 11, 420] that if

$$(1)\quad \sum_{\nu=0}^{n} a_\nu s_{n-\nu}=\frac{n^2}{2}+O(n) \text{ and } a_n\geqq 0 \ (s_n=\sum_{k=1}^{n} a_k),$$

then $s_n=n+o(n)$; but $s_n=n+o(n^{\frac{1}{2}})$ does not have to hold. Avakumović proved [Acad. Serbe Sci. Publ. Inst. Math. 6 (1954), 47–56; MR 16, 239] that (1) implies $s_n=n+O(n^{\sqrt{3}-1+\epsilon})$. The authors now show that (1) implies

$$(2)\qquad s_n=n+O(n^{2/3}(\log n)^{1/3}).$$

If in (1) the condition $a_n\geqq 0$ is weakened to $a_n>-c$, $\limsup_{n\to\infty}|a_n|^{1/n}\leqq 1$, they prove that (2) can still be deduced. They also show that the condition

$$\limsup_{n\to\infty}|a_n|^{1/n}\leqq 1$$

can not be omitted.

The problem whether (1) implies $s_n=n+O(n^{\frac{1}{2}})$ remains open.

P. Erdős (Birmingham).

Juan y Hernandez, Enrique. Infinite determinants and infinite systems of linear equations with infinitely many unknowns. Rev. Mat. Hisp.-Amer. (4) 16 (1956), 15–48. (Spanish)

The author argues that the assumption of normality or absolute convergence of the determinant of a system of linear equations, as given by von Koch and expounded by F. Riesz [Les systèmes d'équations linéaires ⋯, Gauthier-Villars, Paris, 1913], is too restrictive, and he proposes two other classes of determinants: quasi normal (q.n.) and his own brand of absolute convergence (a.c.). An infinite determinant $\Delta=(a_{ik}+\delta_{ik})$ is q.n. if all the series $\sum_{i=1}^{\infty}\binom{i}{k}$ converge absolutely, where $\binom{i}{k}$ is the minor of Δ obtained by omitting the ith row and kth column; and Δ is given by the absolutely convergent series

$$\Delta=\binom{k}{k}+\sum_{i=1}^{\infty}a_{ik}\binom{i}{k}.$$

And Δ is a.c. if $P_1+\sum_{n=2}^{\infty}(P_n-P_{n-1})$ converges, where P_n is the nth "permanent" of Δ; i.e., letting $\alpha_{ik}=|a_{ik}|$, completely expand the finite determinant $(\alpha_{ik}+\delta_{ik})$, $i, k=1, \cdots, n$, and change each minus sign to plus; the resulting new sum is P_n.

Normal and q.n. determinants are also a.c. Numerous properties of q.n. and a.c. determinants are given, and their usefulness indicated in the existence and uniqueness problems for infinite systems of linear equations. Some examples illustrate the theory.

I. M. Sheffer.

See also: Dutta, p. 194; Bateman and Erdős, p. 195; Jerison and Rabson, p. 197; Pandey, p. 200; Rahman, p. 200; Pinkham, p. 209; Lehmann, p. 213; Pettineo, p. 216; Bašeleišvili, p. 246; Nowacki, W. p. 247; Karas, p. 247; Higashi, p. 248; Nowacki and Olesiak, p. 248; Rozovskiĭ, p. 249; Miles, p. 253; Kagan and Yudin, p. 257; Klepikov, p. 259.

Approximations, Orthogonal Functions

Remež, E. Ya. On effective solution of a system of inconsistent linear equations according to Čebyšev's principle of best uniform approximation. Dopovidi Akad. Nauk Ukrain. RSR 1956, 315–320. (Ukrainian. Russian summary)

The author discusses methods of solving systems of inconsistent linear equations

$$F_i(x)\equiv\sum_{j=1}^{n}a_{ij}x_j+b_i=0\ (i=1, 2, \cdots, N)$$

according to Čebyšev's principle of best uniform approximation

$$\max|F_i(x)|=\min\quad(i=1, 2, \cdots, N).$$

After a concise summary of the methods elaborated previously, the author gives a preview of a forthcoming monograph which presents a new effective numerical method of prevailing deviations for the solution of the problem.

S. Kulik (Columbia, S.C.).

Backus, G. E. On the application of eigenfunction expansions of the problem of the thermal instability of a fluid sphere heated within. Phil. Mag. (7) 46 (1955), 1310–1327.

This analysis follows Chandrasekhar's scheme for lowering the order of a high order differential equation [Phil. Mag. (7) 43 (1952), 1317–1329; MR 16, 639]. However instead of solving for the eigenvalues by a variational technique, they are found as the zeros of a certain explicit meromorphic function of a single complex variable. Although this rather clever method (at least to the reviewer) is a bit more tedious than the variational technique for the lowest eigenvalue, it allows bracketing and independent computation of any eigenvalues. Most of the paper is concerned with a discussion of the roots of the meromorphic function. However at the end of the paper the method of attack is described in general terms.

R. C. Di Prima (Los Angeles, Calif.).

Dickinson, D. J.; Pollak, H. O.; and Wannier, G. H. On a class of polynomials orthogonal over a denumerable set. Pacific J. Math. 6 (1956), 239–247.

In a note by Favard [C. R. Acad. Sci. Paris 200 (1935), 2052–2053], a proof is outlined that if a sequence of real polynomials $\{\phi_n\}$ possesses a recursion relation $\phi_{n+1}(x)=(x-a_n)\phi_n(x)-\lambda_n\phi_{n-1}(x)$ with $\phi_0(x)=1$, $\phi_1(x)=x-a_0$, $\phi_{-1}(x)=0$, and $\lambda_n>0$, then the polynomials are orthogonal over some set on the real axis with respect to some weight function. The authors say it seems likely that the set and the weight function can be found with ~~sufficient~~ effort. In particular, they study the case where $a_n=0$ and $\sum\lambda_n$ converges, and show that the polynomials are orthogonal over a denumerable set of points. They find the weight function explicitly in terms of the entire function $E(z)=\lim z^n\phi_n(1/z)$. A set of modified Lommel polynomials illustrates the theory.

R. P. Boas, Jr.

Cremonesi, Aldo. Sull'ortonormalizzazione di un particolare sistema di funzioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 168–171.

The author wishes to orthonormalize the system $e^{-\alpha_1 t}, te^{-\alpha_1 t}, \cdots, t^{p_1-1}e^{-\alpha_1 t}; \cdots; e^{-\alpha_n t}, \cdots, t^{p_n-1}e^{-\alpha_n t}$. He finds that the Laplace transforms of the elements of the orthonormal system can be expressed in a simple form.

R. P. Boas, Jr. (Evanston, Ill.).

Endl, Kurt. Les polynomes de Laguerre et de Hermite comme cas particuliers d'une classe de polynomes orthogonaux. I. Ann. Sci. Ecole Norm. Sup. (3) 73 (1956), 1–13.

A detailed discussion of the polynomials $P_{k,n}(z)$ introduced previously by the author [C.R. Acad. Sci. Paris **241** (1955), 723–724; MR **17**, 149] is given. In particular, it is shown that

$$P_{k,n}(z)=A_{k,n}z^{n_k^*}{}_1F_1(-n_k, (2n_k^*+1)/k; z^k),$$

where $n=n_kk+n_k^*$, $0\leq n_k^*<k$, and $A_{k,n}$ is known and that the system $\{P_{k,n}\}$ is involutive, possessing 2^k involutive forms. *N. D. Kazarinoff.*

Head, J. W. Approximation to transients by means of Laguerre series. Proc. Cambridge Philos. Soc. **52** (1956), 640–651.

Let $F_1(s)$ denote the Laplace transform of a function of the type $F(t)=\sum_1^m A_ne^{b_nt}$, $\Re(b_n)\leq 0$. $F_1(s)$ can be written as a quotient of polynomials in s. Ward [same Proc. **50** (1954), 49–59; MR **15**, 308] has modified a method, due to Tricomi and others, for approximating $F(t)$ by means of series involving Laguerre functions, which sometimes makes it unnecessary to determine the poles of $F_1(s)$. The method is further investigated with reference to conditions for convergence and devices for improving rapidity of convergence. The poles of $F_1(s)$ are assumed to be approximately known. Their determination involves finding complex roots of polynomial equations. A brief discussion of Lin's method [J. Math. Phys. **20** (1941), 231–242; MR **3**, 153] for locating such roots is given.
N. D. Kazarinoff (Ann Arbor, Mich.).

Campbell, Robert. Sur certains procédés de sommation simples relatifs à des séries de polynomes orthogonaux et à des séries de Neumann. C. R. Acad. Sci. Paris **243** (1956), 1007–1009.

In a previous note [same C.R. **243** (1956), 882–885; MR **18**, 125] the author presented an effective procedure for calculating the means (C, k) relative to expansions in series of orthogonal polynomials of various families including those of Hermite and Laguerre. This procedure reduces to the application of a differential operator to a partial sum of the series in question. Here, an analogous procedure is sought for another class of polynomials. Application is made to the case of expansions in series of ultraspherical polynomials and, as it turns out to be possible, to Neumann series. *N. D. Kazarinoff.*

Crum, M. M. On the theorems of Müntz and Szász. J. London Math. Soc. **31** (1956), 433–437.

The author considers a Banach space X which is one of $L^p(0, \infty)$, $p\geq 1$, or else $C_{\cdot 0}(0, \infty)$, the space of continuous functions tending to 0 at ∞. Let $\{\lambda_n\}$ be a sequence of distinct positive numbers, and let S be the set of finite linear combinations of $e^{-\lambda_nx}$. The theorems of the title show that in special cases (A): S is dense in X is equivalent to (B): $\sum \lambda_n/(1+\lambda_n^2)=\infty$. By general theorems, (A) is equivalent to (C): every linear functional on X, vanishing on S, vanishes identically. The author first shows that (B) and (C) are equivalent in general; hence so are (A) and (B). If λ_n are complex with positive real parts, (B) is replaced by (B'): $\sum \Re(\lambda_n)/(1+|\lambda_n|^2)=\infty$. Then (A) and (B') are equivalent if $\{\lambda_n\}$ has at most a finite number of limit points on $\Re(\lambda)=0$, but the proof breaks down otherwise. The author shows that if X is L^p and $1\leq p\leq 2$, then (A) implies (B'), while if X is L^p and $p\geq 2$, or X is $C_{\cdot 0}$, then (B') implies (A). *R. P. Boas, Jr.*

Berg, Lothar. Asymptotische Entwicklung einer Klasse von Integralen. Z. Angew. Math. Mech. **36** (1956), 245–246.

The author states the following theorem, with a sketch of the proof; details are to appear later. Let $f(s)$ be analytic for $\Re s\geq\sigma_1$. With $s=\sigma+i\tau$, let $f^{(\nu+1)}(s)=O(\sigma^{\alpha-1}|f^{(\nu)}(\sigma)|)$, for fixed τ as $\sigma\to\infty$ for $\nu=0, 1, \cdots, 2l$, with $\alpha<\frac{1}{2}$. For some a, let $f^{(2l)}(\sigma+\omega)=O(|f^{(2l)}(\sigma)|)$ for real ω with $|\omega|<a\sigma^{(\alpha+1)/2}$. Let $\int_0^{\infty} t^{s-1}|f(t)|dt$ be finite for $s=\sigma_0$. Then

$$\frac{1}{\Gamma(s)}\int_0^\infty e^{-t}t^{s-1}f(t)dt=\sum_{\nu=0}^{2l-1}\frac{1}{\nu!}\,Q_\nu(s)f^{(\nu)}(s)+O(\sigma^l|f^{(2l)}(\sigma)|)$$

as $\sigma\to\infty$, where $Q_\nu(s)$ is an explicitly given polynomial of degree $[\nu/2]$. *R. P. Boas, Jr.* (Evanston, Ill.).

See also: Climescu, p. 198; Mergelyan, p. 202; Singer, p. 221; Janne d'Othée, p. 242; Soitani, p. 243.

Trigonometric Series and Integrals

Kalašnikov, M. D. On a method of approximation of functions which satisfy Lipschitz conditions by trigonometric polynomials. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 325–329. (Ukrainian. Russian summary)

Let $E_n^{(N)}(KH^{(\alpha)}, x)$ denote the least upper bound of $|f(x)-\sigma_n^N(f, x)|$ over all functions of the class $KH^{(\alpha)}$, $0<\alpha\leq 1$, of functions having period 2π and satisfying Lipschitz condition of order α with a constant K. Here

$$\sigma_n^N(f, x)=(n+1)^{-1}\sum_{m=0}^N T_{n,m}^N(f, x),$$

where

$$T_{n,0}^N(f, x)=\frac{a_0^N}{2},$$

$$T_{n,m}^N(f, x)=\frac{a_0^N}{2}+\sum_{\nu=1}^m (a_\nu^N \cos \nu x+b_\nu^N \sin \nu x) \qquad (m=1, 2, \cdots, n),$$

$$a_0^N=\frac{2}{N}\sum_{k=1}^N f(x_k),\quad a_\nu^N=\frac{2}{N}\sum_{k=1}^N f(x_k)\cos \nu x_k,$$

$$b_\nu^N=\frac{2}{N}\sum_{k=1}^N f(x_k)\sin \nu x_k,$$

$$(\nu=1,\ 2,\ \cdots,\ n;\ x_k=2\pi kN^{-1},\ N\geq 2n+1).$$

The author proves the following theorem: If $\alpha=1$, then $E_n^{(N)}(KN^{(\alpha)}, x)=2K \log n/(n\pi)+O(n^{-1})$. If $0<\alpha<1$, then

$$E_n^{(N)}(KN^{(\alpha)}, x)=\frac{4K}{n^\alpha\omega^{1-\alpha}}\sum_{k=-\infty}^{\infty}\frac{\sin^2 (2k-Nx)}{|12k-Nx|^{2-\alpha}}+o(n^{-\alpha}),$$

provided $\lim n/N=\omega\neq 0$, as $n\to\infty$, but

$$E_n^{(N)}(KN^{(\alpha)}, x)=\frac{2K\Gamma(\alpha)\sin(\frac{1}{2}\alpha\pi n^{-\alpha})}{\pi(1-\alpha)}+o(n^{-\alpha})$$

when $\alpha=0$. *S. Kulik* (Columbia, S.C.).

Sinvhal, S. D. Sur la sommabilité (C, 1) **de la série de Fourier.** Bull. Sci. Math. (2) **79** (1955), 169–173.

F. T. Wang [J. London Math. Soc. **22** (1947), 40–47; MR **9**, 182] has proved that if $\int_0^t \varphi(u)du=o(t/\log(1/t))$ as $t\to 0$, where $\varphi(u)=f(x+u)+f(x-u)-2f(x)$, then the Fourier series of $f(t)$ is summable $(C, 1)$ at $t=x$. The author generalizes this in the following form: if (i) the

integral

$$\xi(t)=\lim_{\epsilon\to 0}\int_\epsilon^t \varphi(u)u^{-1}(\log u^{-1})^\alpha du \quad (0<\alpha<1)$$

exists, (ii) $\theta(t)=\xi(t)/(\log(1/t))^\alpha\to 0$ as $t\to 0$, and (iii) $t\theta(t)$ is of bounded variation in the neighborhood of $t=0$, its total variation in $(0, t)$ being of order $O(t)$, then the Fourier series of $f(t)$ is summable $(C, 1)$ at $t=x$. The condition is of the Young type. {The reviewer [Tôhoku Math. J. 1 (1950), 144–166; MR 11, 656] proved a similar theorem of the Lebesgue type by the same idea.} By constructing an example, the author shows that his theorem is effectively better than Wang's. *S. Izumi* (Sapporo).

Sunouchi, Gen-ichiro. Errata "Cesàro summability of Fourier series". This journal, vol. 5 (1953), 193-210. Tôhoku Math. J. (2) 8 (1956), 147.

Typographical errors are noted and corrected. *P. Civin* (Eugene, Ore.).

Matsumoto, Kishi. Local property of the summability $|R, \lambda_n, 1|$. Tôhoku Math. J. (2) 8 (1956), 114–124.

Let Δ, λ_n, and l_n be such that, when n is sufficiently great, one of the three sets of conditions

(i) $0<\Delta\leqq 1$, $\lambda_n=\exp n^\Delta$, $l_n=1/n^\Delta(\log(n+1))^{1+\epsilon}$;

(ii) $\Delta>0$, $\lambda_n=\exp(\log n)^\Delta$, $l_n=1/(\log(n+1))^{\Delta+\epsilon}$;

(iii) $\Delta>0$, $\lambda_n=\exp(\log\log n)^\Delta$, $l_n=1/(\log\log(n+1))^{\Delta+\epsilon}$

is satisfied. If $\epsilon=0$ and $x<\alpha<\beta<x+2\pi$, then there is a function f, integrable Lebesgue over $\alpha\leqq t\leqq\beta$ and vanishing elsewhere over $x\leqq t\leqq x+2\pi$, which has a trigonometric Fourier series $\sum A_n(t)$ such that the series $\sum l_nA_n(x)$ is not evaluable $|R, \lambda_n, 1)$, that is, $\sum l_nA_n(x)$ is not absolutely evaluable by the Riesz method of type λ_n and order 1. This shows that, when $\epsilon=0$, evaluability $|R, \lambda_n, 1|$ of $\sum l_nA_n(x)$ is not a local property of f. However, if $\epsilon>0$, then such evaluability is a local property. *R. P. Agnew* (Ithaca, N.Y.).

Pyateckiĭ-Šapiro, I. I. On the problem of the uniqueness of the expansion of a function in a trigonometric series. Moskov. Gos. Univ. Uč. Zap. 155, Mat. 5 (1952), 54–72. (Russian)

This is the paper whose supplement [same Zap. 165, Mat. 7 (1954), 79–97, cited hereinafter as PS II] has been reviewed in MR 16, 691, and some of the content is covered in that review. This paper makes several important advances in the uniqueness problem, but also contains some mistakes that are corrected in PS II. The basic error was in assuming that the weak-star topology in the space conjugate to a separable Banach space satisfies the first countability axiom. A closed set E is called a set of multiplicity if there is a trigonometric series converging to zero on the complement of E, but not vanishing identically. Otherwise E is a set of uniqueness. Let W denote the ring of functions with absolutely convergent Fourier series. W is the space conjugate to the space (c_0) of sequences tending to zero.

Theorem. A sufficient condition for E to be a set of uniqueness is the existence of a sequence of functions f_n such that: (i) $f_n\in W$, (ii) $f_n'\in W$, (iii) $f_n(x)=1$ in E, (iv) f_n converges (weak-star) to zero. It is an unsolved problem whether or not condition (ii) may be omitted. In the paper under review it is falsely stated that the conditions are necessary. In PS II it is shown that every closed set of uniqueness is a countable union of closed sets, each of which satisfies conditions (i), (iii) and (iv).

A closed set E is a pure M-set if every non-void portion of E is a set of multiplicity. A "portion" of E is any intersection of E with an open interval. Theorem. E is a pure M-set if and only if there is a weak-star closed ideal S in W such that E is the set of common zeros of the functions in S.

A family of s sequences of positive integers $(n_{k,i})$ $(k=1, 2, \cdots; i=1, 2, \cdots, s)$ is called independent if for any integers (positive or negative) $x_1, \cdots, x_s$ not all zero we have: $\lim|\sum_{i=1}^s x_in_{k,i}|=\infty$ as $k\to\infty$. A set $E\subset[0, 1]$ is of type H^s if there exist s independent sequences, and s intervals $\Delta_1, \cdots, \Delta_s\subset[0, 1]$, such that for every pair (x, k), $x\in E$, k a positive integer, there is an integer j $(1\leqq j\leqq s)$ such that $(n_{k,j}x)\notin\Delta_j$, where (y) is the fractional part of y. Theorem. The sets of type H^s are sets of uniqueness. It was an old conjecture of Rajchman that every closed set of uniqueness was a countable union of sets H^1. The author disproves this by showing that for each s the family H^s contains closed sets that are not countable unions of sets from families of smaller indices.

A set is called M in the restricted sense if there is a positive unit mass on E with Fourier-Stieltjes coefficients tending to zero. Otherwise E is called U in the wide sense. Theorem. A sufficient condition for E to be a set of uniqueness in the wide sense is the existence of a sequence of functions f_n such that: (i) $f_n\in W$, (ii) $\lim\inf f_n(x)>0$ for each x in E, (iii) f_n converges (weak-star) to zero. Every restricted M-set is a set of multiplicity, but the author shows that the converse is false. The example given is incorrect, but in PS II he shows that a counter-example is furnished by fixing any number d, $0<d<\frac{1}{2}$, and taking for E the set of all x admitting a dyadic expansion $x=0.\delta_1\delta_2\cdots$ in which $\delta_1+\cdots+\delta_n\leqq dn$ for all n.

The paper closes with a brief discussion of the possibility of extending the results to general compact abelian groups. *A. Shields* (Ann Arbor, Mich.).

See also: Whiteman, p. 195; Gundlach, p. 195; Kovan'ko, p. 203; Wilde, p. 246; Higashi, p. 248; Nowacki and Olesiah, p. 248; Kączkowski, p. 249; Piątek, p. 250; Levin p. 201.

Integral Transforms

Steinberg, Jacob. Sur les lois de commutation de certaines transformations intégrales. Ann. Scuola Norm. Sup. Pisa (3) 10 (1956), 25–33.

Let $\mathcal{K}$ denote the integral operator defined by $\mathcal{K}f(t)=\int_a^b K(s, t)f(t)dt$, let Δ and ξ be linear differential operators and let D be the operator of differentiation, x the operator which multiplies $f(x)$ by x. If $\mathcal{K}$ is the Laplace transform, it has (for the particular choices $\Delta=-x$, $\xi=D$) the properties $D\mathcal{K}\to\mathcal{K}\Delta$, $x\mathcal{K}\to\mathcal{K}\xi$, where the arrow means that the expression on the left is equal to the expression on the right plus a bilinear form in the derivatives of f and the kernel. The author asks for conditions under which a nondegenerate $\mathcal{K}$ has these two properties, and finds that it is necessary and sufficient that ξ is of order at least 1 and $\Delta\xi-\xi\Delta\equiv 1$. *R. P. Boas, Jr.*

Rathie, C. B. Some properties of generalised Laplace transform. Proc. Nat. Inst. Sci. India. Part A. 21 (1955), 382–393 (1956).

Let $f(p)\doteqdot h(x)$ mean $f(p)=p\int_0^\infty e^{-px}h(x)\,dx$ and let $\phi(p)\underset{k,m}{\overset{v}{=}}h(x)$ mean $\phi(p)=p\int_0^\infty e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}}W_{k,m}(px)h(x)dx$.

The author gives a list of about 20 transform pairs of the second kind, three theorems, and some applications. The first theorem expresses ϕ in terms of f when $\phi(p) \underset{k,m}{\overset{v}{=}} h(x)$ and $f(p) \underset{\sigma,\mu}{\overset{v}{=}} x^{2m-2\mu}h(x)$. The second theorem expresses ϕ in terms of f when $\phi(p) \doteqdot h(x)$, $p^{2-\lambda}h(p) \underset{k,m}{\overset{v}{=}} g(x)$, and $p^{2-\mu}g(p) \doteqdot f(x)$. These are applied to evaluate some integrals involving hypergeometric functions, Bessel functions, and E-functions. The third theorem gives recurrence relations for $\phi(p)$ under change of k and m. As applications the author obtains some recurrence relations for E-functions and Whittaker functions, for example

$$(m-k-\tfrac{1}{2})W_{k,m}(p)= (\tfrac{1}{2}-k-m)W_{k,m-1}(p)+(2m-1)p^{-\frac{1}{2}}W_{k+\frac{1}{2},m-\frac{1}{2}}(p).$$

R. P. Boas, Jr. (Evanston, Ill.).

Pleijel, Arne. Beiträge zur Theorie der Laplace-Stieltjes-Transformationen. Math. Scand. 4 (1956), 147–152.

Let $f(s)=\int_0^\infty e^{-sx}d\alpha(x)$ converge for some $s=\sigma+it$ of positive real part. Then a sufficient condition that $f(s)$ possesses an analytic continuation into $\Re(s)>0$ such that $f(s)=O(t^k)$, $k\geqq 0$, is that

$$\int_0^y e^{-i\tau x}(1-x/y)^\kappa d\alpha(x) \leqq K(1+|\tau|)^k$$

for some $\kappa>k$, some K, and all real τ and positive y. Conversely, if $f(s)$ has the property in question, for each $\kappa>k$ there is a K such that this inequality holds.

R. P. Boas, Jr. (Evanston, Ill.).

Wuyts, P. On the zeros of a Laplace transform. Simon Stevin 31 (1956), 37–46. (Dutch)

The author's main object is to give conditions on $F(t)$ which will ensure that its Laplace transform $f(s)$ has at most a finite number of zeros in a horizontal half-strip. Let F_n denote the nth iterated integral of F with origin 0. The principal theorem states that if F is real in $[0, a]$ and if for some n, $F_n(t)$ does not change sign in $[0, a]$, but is not identically 0, then for any Y and sufficiently large X, $f(x+iy)$ (supposed to have a half-plane of convergence) has no zeros in $x>X$, $|y|<Y$. The author extends this to complex F, where the values of F_n are supposed to lie in an angle of opening less than π. He makes a number of remarks on the distribution of the u-points of a Laplace transform under various hypotheses. *R. P. Boas, Jr.*

Wuyts, P. On the representation of an analytic function by a Laplace-integral. Nieuw Arch. Wisk. (3) 4 (1956), 71–80.

The following conditions are collectively sufficient for the function $f(s)$ to be the Laplace transform in $x=\Re(s)>\mu$ of a continuous $F(t)$ (with the integral defining the transform allowed to be improper at 0): there is a $\lambda<\mu$ such that $f(s)$ is analytic for $x>\lambda$; $\int_{\mu-i\infty}^{\mu+i\infty} e^{tz}f(z)dz=2\pi iF(t)$ converges uniformly (as a principal value) in each interval $0<T'\leqq t\leqq T''$; $\int_{\mu-i\infty}^{\mu+i\infty}(1+|z|)^{-1}|f(z)||dz|$ converges; $f(s)=o(|y|)$ uniformly in $x\geqq\mu$; $f(s)\to 0$ in the angle $|\arg(s-\mu)|\leqq\psi$, where $0<\psi<\pi/2$. *R. P. Boas, Jr.*

Pinkham, R. S. An inversion of the Laplace and Stieltjes transforms utilizing difference operators. Trans. Amer. Math. Soc. 83 (1956), 1–18.

For the Stieltjes transform $f(x)=\int_0^\infty (x+t)^{-1}\phi(t)dt$ the author introduces the inversion operator

$$\lim_{n\to\infty} c_n(-1)^{n-m}\Delta^n x^m f(xy),$$

where $\Delta g(x)=g(1)-g(0)$, $m=[\log n]$, $c_n=(\log n)^{m+1}/m!$. He shows that it inverts the Stieltjes transform under various hypotheses, and also that it leads to a new inversion of the Laplace transform. Some representation theorems are obtained and the moment problem $\mu_n=\int_0^\infty (n+t)^{-1}\phi(t)dt$ is inverted. *R. P. Boas, Jr.*

Porcelli, Pasquale. Note on a Stieltjes type of inversion. Canad. J. Math. 8 (1956), 447–448.

The author proves that if $g(t)$ is of bounded variation on $[0, 1]$ and if $F(z)=\int_0^1 (1+zt)^{-1}dg(t)$ then

$$(-\pi t)^{-1}\Im F(-t^{-1}+iy)\to\tfrac{1}{2}[g'^{+}(t)+g'^{-}(t)]$$

as $y\to 0+$ at any t in $(0, 1)$ at which the right and left hand derivatives exist. *I. I. Hirschman.*

See also: Matsushita, p. 203; Head, p. 207; Berg, p. 207; Gurevič, p. 216; Cotlar, p. 219; Mackie, p. 251; Miles, p. 253; Wichmann and Kroll, p. 260; Šatunov, p. 215.

Ordinary Differential Equations

★Матвеев, Н. М. Методы интегрирования обыкновенных дифференциальных уравнений. [Matveev, N. M. Methods of integration of ordinary differential equations.] Izdatel'stvo Leningradskogo Universiteta, 1955. 655 pp. 22 rubles.

This text-book is designed for an introductory course in the theory of ordinary differential equations. Although more detailed than other such texts, the book has no outstanding features. The usual first-order equations are studied in the first two chapters. The general problem of solving systems of first-order equations and of solving nth order equations is discussed in Chapters III and IV. The Picard-Lindelöf theorem is presented in Chapter V. Singular points and singular solutions are also discussed in this chapter. The next six chapters are devoted to a study of linear equations. Linear equations of nth-order are studied first. This is then followed by a study of linear systems of first-order equations. Finally, matrix methods are introduced but are limited to their application to homogeneous linear equations. {This order of presentation is somewhat illogical and is certainly inefficient. Perhaps this is indicative of the level at which this course in taught.} The last chapter is a brief look at first-order linear partial differential equations. *J. P. LaSalle.*

Azbelev, N. V. On limits of applicability of the theorem of Čaplygin on differential inequalities. Mat. Sb. N.S. 39(81) (1956), 161–178. (Russian)

The author gives a generalization of the method of Čaplygin for the solution of differential equations. The principal result is as follows.

Let (1) $y^{(n)}=f[y]$ be a differential equation with the initial conditions (2) $y^{(k)}(x_0)=y_0^{(k)}$ having a single solution. Here and in the following $k=0, 1, \cdots, n-1$; $f[y]=f(x, y, y', \cdots, y^{(n-1)})$ is a continuous function single valued for $x_0\leqq x\leqq X$, $a_k\leqq y^{(k)}\leqq b_k$, where $a_k<y_0^{(k)}<b_k$. Let $f[y]$ satisfy the following inequalities,

$$f[y_1]-f[y_2]>\sum_{k=0}^{n-1}(y_1^{(k)}-y_2^{(k)})q_k \quad (x_0<x\leqq X),$$

and any pair of values $y_1^{(k)}$ and $y_2^{(k)}$, if $b_k>y_1^{(k)}>y_2^{(k)}>a_k$ and $q_k=q_k(x)$ are functions continuous in the

interval $[x_0, X]$. Let a function $z=z(x)$, which has n continuous derivatives, be called upper (lower) function of comparison of the problem (1)—(2) if: 1) in the interval (x_0, X) the following inequalities are satisfied: $a_k \leqq z^{(k)} \leqq b_k$ and $z^{(n)} \geqq f[z]$ ($z^{(n)} \leqq f[z]$); 2) the differences $\alpha_k = z^{(k)}(x_0) - y_0^{(k)}$ are nonnegative (non-positive); 3) when $\alpha_k=0$, the aggregate of zeros of the difference $z^{(n)}-f[z]$ has no inner points. Then the author proves that there exists an interval (x_0, x^*), which depends only on equation (1), the domain in which equation (1) is considered, and the initial differences α_k, in which $z^{(k)} > y^{(k)}$ ($z^{(k)} < y^{(k)}$) for $k=0, 1, \cdots n-1$, if all $\alpha_k=0$, but for $k=0, 1, \cdots l$, if l is the largest number for which $\alpha_l \neq 0$. *S. Kulik.*

Čečik, V. A. On a certain class of systems of ordinary differential equations with singularity. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 784–786. (Russian)

The author investigates the system of differential equations

$$(1) \qquad y_k' = f_k(x, y_1, y_2, \cdots, y_n) \quad (k=1, \cdots, n)$$

with initial conditions

$$(2) \qquad \lim_{x \to 0} y_k(x) = 0 \quad (k=1, \cdots, n).$$

The right hand members of (1) are assumed to be continuous functions of the variables for $x>0$. For $x=0$ they may be unbounded. Such systems are called singular. Here the author considers a system of singular differential equations, where the right hand members are not bounded by summable functions. It is also assumed that the functions f_k and f_k' are continuous in the region

$$D: 0 < x \leqq b, \ |y_i| \leqq a \quad (i=1, \cdots, n).$$

The following theorems are proven:

Theorem 1. Let the following inequalities be satisfied in region D:

$$1) \quad |f_k(x, y_1, \cdots, y_{k-1}, 0, y_{k+1}, \cdots, y_k)| \leqq \varphi(x), \qquad (k=1, \cdots, n);$$

$$2) \quad f_{k,y_s}'(x_1, y_1, \cdots, y_n) \leqq \varphi(x) \quad (k=1, \cdots, l),$$

where $\varphi(x)$ is a continuous summable function, and $0 \leqq l \leqq n$;

$$3) \quad f_{k,y_s}'(x, y_1, \cdots, y_n) \geqq \bar{\varphi}_k(x) \quad (k=l+1, \cdots, n),$$

where $\bar{\varphi}_k(x)$ are positive non-summable functions. Then system (1) has at least one solution satisfying conditions (2).

Theorem 2. Let the condition of theorem 1 be satisfied and let $l<n$. Then the solution of problems (1), (2) is not unique. Many solutions in this case depend on $n-l$ parameters.

Theorem 3. Let the conditions of theorem 1 be satisfied and $l=n$. In the region D let there exist derivatives f_{k,y_i}' ($i=1, \cdots, n$), which for $i \neq k$ satisfy the inequalities

$$|f_{k,y_i}'(x, y_1, \cdots, y_n)| \leqq \varphi(x).$$

Then the system (1) has a single solution which satisfies the zero initial conditions (2).

The above results when applied to the single equation

$$(5) \qquad y' = f(x, y)$$

lead to the following theorems.

Theorem 4. If a continuous differentiable function $\varphi(x)(\varphi(0)=0)$ exists, such that $f(x, \varphi(x))$ is summable; then, if $f_y'(x, y)$ has a summable function as an upper bound, equation (5) has a single solution satisfying the zero initial condition.

If $f_y'(x, y)$ is bounded below by a positive non-summable function, then equation (5) has a continuum of solutions.

Theorem 5. Let there be a continuous differentiable function $\bar{\varphi}(x)$ in the interval $[0, \varepsilon]$ $(0<\varepsilon\leqq b)$, such that 1) $f_y'(x, y)>0$ for $y>\bar{\varphi}(x)$, 2) $f(x, y) > \max \bar{\varphi}'(x)$, 3) $\bar{\varphi}(0)=0$, 4) $\int_0^\varepsilon f(x, \bar{\varphi}(x))dx = +\infty$. Then equation (5) has no solution which satisfies the initial condition $y(0)=0$.

For the general case the following theorem is stated:

Theorem 6. Let the conditions of theorem 3 be satisfied and let

$$y_k = y_k(x, x_0, y_{1,0}, \cdots, y_{n,0}) \quad (k=1, \cdots, n)$$

be the solution of system (1) which passes through the point $x=x_0$, $y_k=y_{k,0}$ ($k=1, \cdots, n$). Then the functions $y_k=y_k(x, x_0, y_{1,0}, \cdots, y_{n,0})$ are continuous functions of the arguments in the region $0<x_0\leqq x\leqq b$, $|y_{k,0}|\leqq a$; $x_0=0$, $y_{n,0}=0$ ($k=1, \cdots, n$).

S. D. Zeldin (Cambridge, Mass.).

Zindler, R. E. A note on L. E. Reizin's paper "Behavior of integral curves of systems of three differential equations near a singular point." Proc. Amer. Math. Soc. **7** (1956), 283–289.

In the first part of the paper the author corrects a careless statement of hypotheses in Theorem 2 of a paper of Reĭzin' [Latvijas PSR Zinātņu Akad. Vēstis **1951**, no. 2(43), 333–346; MR **15**, 311; **17**, 482]. In the second part an alternative form of the hypotheses is proposed. {The reviewer finds the justification given insufficient.}

W. Kaplan (Ann Arbor, Mich.).

Iwano, Masahiro. Sur les points singuliers d'une équation différentielle ordinaire linéaire du $n^{\text{ième}}$ ordre. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1956), 343–351.

The author considers the problem of the asymptotic series developments for solutions of

$$x^s y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_n(x) y = 0,$$

where s is a positive integer and the $a_j(x)$ are analytic in a neighborhood of the origin. Use is made of the results for systems due to Hukuhara [Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **2** (1942), 125–137; MR **9**, 92]. Results due to Perron are also obtained.

N. Levinson (Cambridge, Mass.).

Taam, Choy-tak. Asymptotic relations between systems of differential equations. Pacific J. Math. **6** (1956), 373–388.

Solutions of the vector equation $y'=A(x)y+f(x, y)$ are related to the solutions of $u'=A(x)u$ through an integral equation in the usual way. Conditions are given which ensure that $y/u=1+o(1)$, or that $y-u=o(1)$, as $x\to\infty$. Phragmén-Lindelöf theorems are used to establish similar results for x and y complex and f linear in y. Similar results are also given for second order equations.

T. E. Hull (Vancouver, B.C.).

Zlámal, Miloš. Über asymptotische Eigenschaften der Lösungen der linearen Differentialgleichung zweiter Ordnung. Czechoslovak Math. J. **6(81)** (1956), 75–93. (Russian summary)

In the reviewer's book, "Stability theory of differential equations" [McGraw-Hill, New York, 1953, p. 163; MR **15**, 794] a number of heuristic observations were made concerning the asymptotic behavior of the solutions of

the equations $u''+a(t)u'+u=0$ with $a(t)\to\infty$ as $t\to\infty$, and of $\varepsilon(t)u''+u'+u=0$ with $\varepsilon(t)\to 0$ as $t\to\infty$. In this paper, a detailed treatment of these problems is given, together with a study of the asymptotic behavior of the solutions of $u''+f(t)u=0$, where $f(t)\to\infty$ as $t\to\infty$.

R. Bellman (Santa Monica, Calif.).

Massera, José L. Qualitative study of the equation $u''^2=u+u'$. Bol. Fac. Ingen. Agrimens. Montevideo 5 (1956), 339–347 = Fac. Ingen. Agrimens. Montevideo. Publ. Didact. Inst. Mat. Estadist. 3 (1956), 1–10. (Spanish)

The problem of determining whether or not the equation $u'+u=(u'')^2$ has a solution which approaches zero like e^{-t} as $t\to\infty$, for suitably chosen values of $u(0)$ and $u'(0)$ was proposed by the reviewer [Bull. Amer. Math. Soc. 61 (1955), 192]. In this paper, the author furnishes a complete solution to the problem, with an affirmative answer, by means of a detailed study of the family of solutions of the above differential equation. *R. Bellman.*

Grobman, D. M. Asymptotic behavior of solutions of non-linear systems that deviate little from linearity. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 571–574. (Russian)

The author studies the asymptotic behavior of solutions of $dx/dt=Ax+f(t,x)$ under the assumption that $|f(t,x)-f(t,y)|\leq g(t)|x-y|$, and various additional assumptions concerning $g(t)$. *R. Bellman.*

Beesack, P. R.; and Schwarz, Binyamin. On the zeros of solutions of second-order linear differential equations. Canad. J. Math. 8 (1956), 504–515.

Let $M(y)$ be an even, positive, continuous function on $(-1,1)$ such that $(1-y^2)^2M(y)$ is non-increasing on $(0,1)$. Let $q(z)$ be regular, $|q(z)|\leq M(|z|)$, in $|z|<1$. Suppose that there is a solution of the real differential equation

$$y''+M(y)y=0$$

which has two consecutive zeros $-a$ and a, $0<a<1$. The authors prove that, for any (non-trivial) solution of the complex differential equation

$$u''+q(z)u=0$$

which vanishes at z_1 and z_2 in $|z|<1$, $\|[z_1,z_2]\|\geq\|[-a,a]\|$. ($\|[k,j]\|$ is the non-Euclidean distance between k and j in the sense of the Klein-Poincaré hyperbolic geometry.) This result generalizes a recent theorem of Nehari [Proc. Amer. Math. Soc. 5 (1954), 700–704; MR 16, 232]. The proof of this theorem consists of two parts: the first part follows the method used by Nehari and the second part makes use of a result whose proof depends on a lower bound of the least positive eigenvalue of certain real differential system. *C. T. Taam* (Washington, D.C.).

Barrett, John H. Matrix systems of second order differential equations. Portugal. Math. 14 (1956), 79–89.

This paper deals with a linear second order matrix differential equation

$$(*)\qquad (P(x)Y')'+Q(x)Y=0,$$

where on a given interval $a\leq x<\infty$ the $n\times n$ matrices $P(x)$, $Q(x)$ have real-valued continuous elements. Under the assumption that P, Q are symmetric, and P is positive definite on $a\leq x<\infty$, the author establishes theorems on the asymptotic character of solutions of (*) that are generalizations of results of E. Hille [Trans. Amer. Math. Soc. 64 (1948), 234–252; MR 10, 376] for scalar equations, and a sufficient condition for oscillation that is supplementary to results of R. L. Sternberg [Duke Math. J. 19 (1952), 311–322; MR 14, 50]. For an equation (*) in which the coefficient matrices are not required to be symmetric there are presented criteria which insure that all solutions remain bounded as $x\to\infty$. *W. T. Reid.*

Kostomarov, D. P. Formal solutions of systems of linear differential equations by expansion into normal and and subnormal series. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 1011–1013. (Russian)

K. Latyševa has shown earlier [Dissertation, Kiev Gos. Univ., 1952] that the differential equation

$$\omega^{(n)}+p_1(z)\omega^{(n-1)}+\cdots+p_n(z)\omega=0,$$

$$p_k(z)=\sum_{n=0}^{\infty}p_k^{(m)}z^{\pi_k-m}\quad(\pi_k\text{—integers})$$

has a fundamental system of solutions in the form of normal and subnormal series of the type introduced by Poincaré [Acta Math. 8 (1886), 295–344].

In this paper the author gives a similar discussion in terms of formal series for systems of linear differential equations. His principal theorem is that the system

$$\frac{d\omega_i}{dz}=\sum_{j=1}^{n}a_{ij}(z)\omega_j,\ a_{ij}(z)=\sum_{m=0}^{\infty}a_{ij}^{(m)}z^{s-m/q}\ (\delta,\ q\text{—integers})$$

has a formal fundamental system of subnormal solutions (defined by a specific series). The discussion involves transforming the system of equations into a new one by a change of variables; and by studying the characteristic values of the matrix of the new system, the author asserts that the formal existence as well as the structure of the solutions can be established. [For detailed proofs the author refers to his Dissertation, Moskow Gos. Univ., 1956.] *S. D. Zeldin* (Cambridge, Mass.).

Cartwright, M. L. On the stability of solutions of certain differential equations of the fourth order. Quart. J. Mech. Appl. Math. 9 (1956), 185–194.

The author considers first the linear equation with constant coefficients

$$(1)\qquad x^{(4)}+a_1x^{(3)}+a_2\ddot{x}+a_3\dot{x}+a_4x=0$$

or the equivalent system

$$(2)\qquad \dot{x}=y,\ \dot{y}=z,\ \dot{z}=w,\ \dot{w}=-a_1w-a_2z-a_3y-a_4x.$$

For (2) she gives the Lyapunov function V_B, such that $\dot{V}_B=-ky^2$, $k>0$ as a sum of four linear squares, all with positive coefficients (given explicitly) if the characteristic roots have negative real parts.

Taking now

$$(3)\qquad x^{(4)}+a_1x^{(3)}+a_2\ddot{x}+a_3\dot{x}+f(x)=0$$

with associated system (3') analogous to (2) she forms ($\cdots$ means: sum of positive squares)

$$V_B=\cdots+a_3A_4(x)y^2+2a_1a_3A_3F(x)-a_1^2f^2(x)$$
$$A_3=a_1a_2-a_3$$
$$A_4=a_1a_2a_3-a_3^2-a_1^2f^1(x)$$
$$F(x)=\int_0^x f(x)dx.$$

The coefficients A_3, and A_4 when $f=a_4x$, are the noted principal minors in the characteristic matrix.

Theorem. If a_1, a_2, $a_3>0$, $f(0)=0$, $f'(x)>0$ and $f''(x)$ is

continuous and $A_4(x)\geqq\delta>0$ for all x considered, finally $F(x)\to\infty$ with $|x|$, then whatever $V_0>0$, $V_B(x,y,z,w)<V_0$ defines a domain D_0 of the space x, y, z, w. The solution of (3′) through $P_0\in D_0$ tends to the origin as $t\to+\infty$, provided that in D_0: $|f''(x)y|<\delta/a_1$.

An analogous treatment is given for

$$x^{(4)}+a_1x^{(3)}+\psi(\dot{x})\ddot{x}+a_3\dot{x}+a_4x=0$$

and a similar theorem is proved. The condition relating to D_0 is now: $a_3|\psi'(y)z|<\delta$, where one imposes:

$$A_4(y)=a_1a_3\psi(y)-a_3^2-a_1^2a_4\geqq\delta>0.$$

Similar results were obtained for the order three by Barbašin [Prikl. Mat. Meh. **16** (1952), 629–632; MR **14**, 376] and Šimanov [ibid. **17** (1953), 369–372; MR **14**, 1087] who likewise used adequate functions V. *S. Lefschetz.*

Lyaščenko, N. Ya. On a theorem on complete separation of a linear homogeneous system of ordinary differential equations and some properties of the separation matrix. Ukrain. Mat. Ž. 7 (1955), 403–418. (Russian)

In previous papers [Dokl. Akad. Nauk SSSR (N.S.) **97** (1954), 965–967; Ukrain. Mat. Ž. 7 (1955), 47–55; MR **16**, 247, 1110], the author considered the problem of transforming a given linear differential equation into two independent linear differential equations of lower order. In this paper, he continues this study, with application to the case where the coefficient matrix is continuous.

R. Bellman (Santa Monica, Calif.).

Breus, K. A. On the solution of linear differential equations with rapidly varying periodical coefficients. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 997–1000. (Russian)

Es sei $x=x(t)$ eine n-dimensionale Vektorfunktion, $A(\omega t)$ eine reelle symmetrische Matrix, stetig und periodisch mit der Periode $T=2\pi/\omega$ und ω ein grosser Parameter. Den Inhalt der Arbeit bildet der Beweis, dass die Fundamentalmatrix des Systems $dx/dt=A(\omega t)x$ eine analytische Funktion des kleinen Parameters $\varepsilon=1/\omega$ ist. Die Methode des Beweises ermöglicht eine effektive Konstruktion. *M. Zlámal* (Brno).

Derwidué, L. Une question de stabilité. Z. Angew. Math. Mech. **36** (1956), 246–248.

The author considers the system of differential equations

$$(1)\qquad \sum_{k=1}^{n}(a_{jk}^0y_k^{(m)}+a_{jk}^1y_k^{(m-1)}+\cdots+a_{jk}^my_k)=0 \qquad (j=1,2,\cdots,n),$$

where the a_{jk}^i are constants. The boundedness of the solutions of (1) depend upon the roots of the characteristic equation

$$(2)\qquad |\alpha_0r^m+\alpha_1r^{m-1}+\cdots+\alpha_m|=0,$$

where each $\alpha_i=[a_{jk}^i]$ is an $n\times n$ matrix. Theorem. If $m=2$, α_0 is positive definite, α_2 is positive semi-definite, and $\alpha_1=d+g$, where d is hermitian and g is skew hermitian; then (A) if d is positive definite all the roots of (2) have real parts negative, or (B) if d is positive semi-definite or zero (2) has only roots with real parts negative or zero and, if the coefficients of (1) are real, a p-triple purely imaginary root lowers the rank of the left member of (2) by p. In both cases (A) and (B), the solutions of (1) are bounded. For $m>2$, the author shows that the characteristic roots of (2) may be found by solving a system of equations each of degree m if the matrices α are permutable. *J. K. Hale* (Albuquerque, N.M.).

Halanay, A. Solutions presque-périodiques des systèmes d'équations différentielles non linéaires. Com. Acad. R. P. Romîne **6** (1956), 13–17. (Romanian. Russian and French summaries)

Suivant le résumé donné par l'auteur: „On considère le système $dx/dt=X(x,t)$, où $X(x,t)$ est presque-périodique par rapport à t, uniformément par rapport à x. Si le système admet une solution limitée $u(t)$, $|u(t)|<M$ asymptotiquement stable, uniformément par rapport au domain $|x|<M$, il admet une solution presque-périodique. A l'aide de ce théoreme général, on obtient les conditions d'existence des solutions presque-périodiques pour le système $dx/dt=Ax+f(x,t)$." *M. Zlámal* (Brno).

Reissig, R. Über die Existenz periodischer Lösungen für Differentialgleichungen 2. Ordnung mit einem Störungsglied. Z. Angew. Math. Mech. **36** (1956), 256–257.

By using the Brouwer fixed-point theorem, the author proves there is at least one periodic solution of the system of equations

$$x'=v,\ v'=e(t)-g(x)-f(x,v)\cdot v$$

if the equation has a unique solution and the following conditions are satisfied: $e(t)$ is periodic with period L; $\max e(t)=M$, $\min e(t)=-M$; $f(x,v)\geqq0$ for $|x|\geqq a$ and $|v|\geqq b$; $\liminf f(x,v)=-f\leqq0$ if $|x|\leqq a$ or $|v|\leqq b$; $g(x)\geqq M+b\cdot f+\delta$, $\delta>0$, for $x\geqq a$; $g(x)\leqq -M-b\cdot f-\delta$ for $x\leqq -a$; $M-g(x)-f(x,v)\cdot v\leqq0$ for $x\leqq -a$ and $v\geqq c\geqq b$; $-M-g(x)-f(x,v)\cdot v\geqq0$ for $x\geqq a$ and $v\leqq -c$. *J. K. Hale.*

Acrivos, Andreas. The transient response of stagewise processes. J. Soc. Indust. Appl. Math. **4** (1956), 1–19.

In this paper, the author characterizes a general stagewise system (e.g. distillation columns, absorption columns, stirred tank reactors) in such a way that the question of the stability of the stationary state can be treated by mathematical methods. He shows that the variations from the stationary state satisfy a non-linear matrix differential equation $x'=Ax+f(x)$, where A is a constant matrix. Necessary and sufficient conditions for the characteristic roots of the matrix A to have real parts negative are then formulated in terms of the Hurwitz-Routh determinants. The transient response of a stable system to variations in the input is also discussed and numerical examples are given. *J. K. Hale.*

Volpato, Mario. Sull'esistenza di soluzioni periodiche per equazioni differenziali ordinarie del secondo ordine. Rend. Sem. Mat. Univ. Padova **25** (1956), 371–385.

It is shown that if the following conditions (1), (2), (3) are satisfied, the equation $\ddot{x}+P(t,x,\dot{x})x=Q(t,x,\dot{x})$ possesses a periodic solution with the period ω. (1) P and Q are real continuous functions of their real arguments, and they are periodic with respect to t with the common period ω. (2) Q is bounded. (3) There exist functions $p_1(t)$ and $p_2(t)$, periodic with the period ω, such that $0\leqq p_1(t)\leqq P(t,x,\dot{x})\leqq p_2(t)$, and

$$\omega\int_0^\omega p_2\,dt\leqq4.$$

The proof depends upon theorems concerning the existence of fixed points of transformations in function spaces. *L. A. MacColl* (New York, N.Y.).

Karaseva, T. M. On the expansion of arbitrary functions in series of eigenfunctions of a boundary-value problem. Har'kov. Gos. Univ. Uč. Zap. 29=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) 21 (1949), 59–75. (Russian)

The system $L(y)-\lambda Ry=0$ is considered on $a\leq x\leq b$. Here $L(y)=-(Ay'+By)'+B^*y'+Dy$, where y is an n-vector and A, B, D and R are $n\times n$ matrix functions of x on $a\leq x\leq b$. Moreover $A=A^*$, $D=D^*$ and R is Hermitian positive except at a finite number of points where it may be non-negative. There are also $2n$ linear homogeneous boundary conditions involving y and $Ay'+By$ at a and b. The expansion theorem for the self-adjoint problem is considered. References are made to the well-known work of M. Morse and to results of M. G. Krein. *N. Levinson.*

Lehmann, N. Joachim. Eine Integraldarstellung für selbstadjungierte Randwertaufgaben (einschliesslich einer Theorie der natürlichen Eigenwertprobleme). Math. Nachr. 14 (1955), 129–156.

This paper deals with the application of earlier work of the author [Math. Nachr. 5 (1951), 139–160; 9 (1953), 45–50; MR 13, 247; 14, 765] on linear integro-differential equations to real self-adjoint boundary problems involving the differential equation

$$\sum_{\nu=0}^{m}(-1)^{\nu}(m_{\nu}(x)y^{(\nu)})^{(\nu)}=\lambda\sum_{\nu=0}^{n}(-1)^{\nu}(n_{\nu})(xy^{(\nu)})^{(\nu)},$$

and two-point boundary conditions that contain linearly the characteristic parameter λ. Following the discussion of attendant results on the reduction of the boundary conditions to special form, and the character of certain associated bilinear functionals on a suitable linear function space, the author shows that the considered boundary problem is equivalent to an integro-differential equation of the sort considered in his earlier papers cited above. For boundary problems possessing certain "definiteness" properties there are established various results on the existence of proper values, expansions for the Green's function and certain of its partial derivatives, and expansion theorems for an associated class of admissible functions. *W. T. Reid* (Evanston, Ill.).

Glasko, V. B. Some problems on characteristic values, involving a small parameter. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 767–769. (Russian)

The authors considers the characteristic value problem for a differential equation of the type

$$\mu\frac{d^2}{dx^2}\left[p_2(x)\frac{d^2y}{dx^2}\right]-\frac{d}{dx}\left[p_1(x)\frac{dy}{dx}\right]+p_0(x)y=\lambda\rho(x)y$$

with $x\in I=\{x|x\in[x_1,x_0)$ or $x\in(x_0,x_2]\}$, $[y]_{x_0}=[p_1y']_{x_0}=[p_2y'']_{x_0}=[(p_2y'')']_{x_0}=0$, and with boundary conditions $y(x_s)=0$, $y^{(\sigma)}(x)_s=0$, $s=1, 2$ and σ either 1 or 2. He obtains the behavior of the characteristic values and functions as $\mu\to 0$ under the following three assumptions: (i) p_2 and $p_1\in C^{(4)}$, $p_0\in C^{(2)}$, $\rho\in C^{(0)}$ when $x\in I$; (ii) at $x_0p_1=0$; the remaining coefficients and their derivatives (those which are involved) have discontinuities of the first kind at x_0; (iii) $p_2(x)$, $p_1(x)$, $\rho(x)\geq k_0>0$, $p_0(x)\geq 0$.

Under these conditions the characteristic values and functions $\lambda_{m\sigma}(\mu)$ and $y_{m\sigma}(x,\mu)$ converge, as $\mu\to 0$, to $\lambda_m^{(0)}(\ )$ and $u_m(x)$ which solve the problem

$$-\frac{d}{dx}\left[p_1(x)\frac{du}{dx}\right]+p_0(x)u=\lambda^{(0)}\rho u,$$

$$[u]_{x_0}=[p_1u']_{x_0}=0,\ u(x_s)=0,\ s=1, 2.$$

However, the convergence is not uniform with respect to m and worsens with increasing m. Explicit forms for $\lambda_{m\sigma}(\mu)$ and $y_{m\sigma}(x,\mu)$ are given. The author uses the method of successive approximations. Because of the nature of the boundary conditions, perturbation theory does not apply. No proofs are presented. *N. D. Kazarinoff.*

See also: Mishoe and Ford, p. 205; Moses, p. 216; Balescu, p. 245; Sparenberg, p. 246; Schmieden und Müller, p. 252; Bopp, p. 259.

Partial Differential Equations

Hornich, Hans. Über Schwingungen mit periodischer Störung und Lösung. Monatsh. Math. 60 (1956), 223–230.

The author considers the equation

$$(1)\qquad \partial^2u/\partial\sigma\partial\tau=Q(\sigma,\tau)\ (-\infty<\sigma,\tau<+\infty),$$

$$Q(\sigma,\tau)=\sum_{n,m\geq 0}a_{nm}\cos[\tau(n+\lambda m)+\sigma(n-\lambda m)]$$

$$(a_{00}=0,\ \sum|a_{nm}|<+\infty).$$

He proves there is a periodic solution of (1) if λ is irrational and either (a) $\liminf|\lambda^2m^2-n^2|>0$, m, n integers ≥ 0, $m+n>0$, or (b) $\liminf|\lambda^2m^2-n^2|=0$ and

$$A=\sum|a_{nm}(\lambda^2m^2-n^2)^{-1}|<+\infty.$$

Furthermore, explicit expressions are given for the solution and, in particular, in case (b), $u(0,\tau)-u(0,0)=B(\tau)$, where

$$B(\tau)=\sum[a_{nm}(\lambda^2m^2-n^2)^{-1}]\cdot[\cos\{\tau(m\lambda+n)\}-1].$$

If $A=+\infty$, then the series $B(\tau)$ diverges except on a set of measure zero which includes the dense set of points $\tau=\pi n'+\pi m'\lambda^{-1}$, n', m' arbitrary integers. He then proves that, if $A=+\infty$, then a continuous periodic solution of (1) is possible only when the function $B(\pi n'+\pi m'\lambda^{-1})$ is uniformly continuous on the set of points $\pi n'+\pi m'\lambda^{-1}$. If λ is rational, there need not be a periodic solution of (1) for every $Q(\sigma,\tau)$. *J. K. Hale* (Albuquerque, N.M.).

Bochenek, K. Equation $(\nabla l)^2=1$ in the complex domain. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 119–123.

Complex-valued solutions of a Cauchy problem for the 2-dimensional Eiconal equation $(\nabla l)^2=1$ are obtained under fairly general assumptions using a complete integral $l=ax+by+c$, where a, b, c are complex numbers such that $a^2+b^2=1$. *G. L. Walker.*

Pawelski, W. Appréciation du domaine d'existence de l'intégrale d'une équation aux dérivées partielles du premier ordre, dans le cas de variables complexes. Ann. Polon. Math. 2 (1955), 37–55.

The author proves that a single nonlinear real analytic first order equation for a single unknown of a number of real variables with prescribed real analytic initial values has an analytic solution in a domain whose width depends only on bounds for the coefficients, initial values and their first and second derivatives. *P. D. Lax.*

Pawelski, W. Appréciation du domaine d'existence de l'intégrale d'un système involutif d'équations aux dérivées partielles du premier ordre. Ann. Polon. Math. 2 (1955), 29–36.

The author proves that a compatible system of m first

order nonlinear equations for a single unknown function of $m+n$ variables has a unique solution with prescribed values on an n-dimensional manifold. The equations and prescribed values are assumed to be twice differentiable, the solution constructed is shown to be once differentiable; the size of the domain of existence depends on the size of the second derivatives. *P. D. Lax.*

Rutman, M. A. On the stability of solutions of certain systems of linear differential equations with variable coefficients. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 770–773. (Russian)

The stability theory of linear differential equations in Banach space developed previously [same Dokl. (N.S.) **101** (1955), 217–220, 993–996; MR **16**, 1126, 1113] is extended to partial differential equations with variable coefficients. Let $\tilde{E}$ be the set of continuous functions $x(t_1, \cdots, t_n)$, $0 \leq t_k < \infty$ $(k=1, \cdots, n)$, taking values in a complex Banach space $\tilde{\mathcal{E}}$. Let the equation

$$\frac{\partial^n y}{\partial t_1 \cdots \partial t_n} - A(t_1, \cdots, t_n) y = x_0(t_1, \cdots, t_n)$$

have the boundary conditions: $y(0, t_2, \cdots, t_n) = x_1(t_2, \cdots, t_n)$, $y(t_1, 0, \cdots, t_n) = x_2(t_1, t_3, \cdots, t_n)$; $\cdots$; $y(t_1, t_2, \cdots, 0) = x_n(t_1, t_2, \cdots, t_{n-1})$, where $y, x_0 \in \tilde{E}, x_k (k=1, \cdots, n)$ is a continuous function from the indicated variables into $\tilde{\mathcal{E}}$, and $A(t_1, \cdots, t_n)$ is a continuous function taking values in $T(\tilde{\mathcal{E}})$, the space of bounded linear transformation of $\tilde{\mathcal{E}}$ into itself. Suppose the set $\{A(t_1, \cdots, t_n)\}$ is a compact set in $T(\tilde{\mathcal{E}})$. Assume also that given $\varepsilon > 0$, there is an $M > 0$ such that if $\sum_{j=1}^n t_j' > M$, $\sum_{j=1}^n t_j'' > M$ and $\sum_{j=1}^n |t_j' - t_j''| \leq 1$, then $\|A(t_1', \cdots, t_n') - A(t_1'', \cdots, t_n'')\| < \varepsilon$. Let

$$\alpha(t_1, \cdots, t_n) = \max_{\lambda \in S_{A(t_1, \cdots, t_n)}} \operatorname{Re} \lambda^{1/n},$$

where S_A is the spectrum of A, and let

$$\alpha_\omega = \limsup_{\sum t_j \to \infty} \alpha(t_1, \cdots, t_n).$$

Conclusion: if for some $\alpha > \alpha_\omega$,

$$\limsup e^{-\alpha \Sigma t_j} \|x_0(t_1, \cdots, t_n)\| < \infty,$$
$$\limsup e^{-\alpha \Sigma t_j} \|x_k(t_1, \cdots, t_{k-1}, t_{k+1}, \cdots, t_n\| < \infty,$$

$(k=1, \cdots, n)$, then $\limsup e^{-\alpha \Sigma t_j} \|y(t_1, \cdots, t_n)\| < \infty$, where these upper limits are all taken as $\sum t_j \to \infty$. Two similar theorems for more complicated equations are stated.

J. Cronin (New York, N.Y.).

Fage, M. K. Differential equations with purely mixed derivatives and a principal term. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 780–783. (Russian)

The author proves existence and uniqueness of a solution of the characteristic initial value problem and of a Cauchy problem for the equation

$$\frac{\partial^n u}{\partial x_1 \cdots \partial x_n} + \sum_{k=0}^{n-1} \sum_{1 \geq i_1 < \cdots < i_k \leq n} p_{i_1 \cdots i_k}(x_1, \cdots, x_n) \times \frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} = f(x_1 \cdots, x_n).$$

The former result is established by the method of Picard, the latter by an extension of the method of Riemann.

R. Finn (Pasadena, Calif.).

Mitrinović, Dragoslav S. Compléments au traité de Kamke. IV. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. **11** (1956), 7–10. (Serbo-Croatian summary)

[For parts I–III see MR **17**, 1086; **18**, 37]. Conditions are given that the equation $xFdy + yGdx = 0$, where F and G are quadratic in x and y, have an integrating factor of the form $M(x^p y^q)$, p and q being constants.

E. Pinney.

Yanenko, N. N. Progressive waves of systems of quasilinear equations. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 44–47. (Russian)

The author calls a solution $u_i x(_1, \cdots, x_m)$ of the system of quasilinear equations

$$\sum_{i,j} a_{ikj}(u_1, \cdots, u_m) \frac{\partial u_j}{\partial x_i} = 0 \quad (i, j, k = 1, \cdots, m)$$

a progressive wave of the first kind when it can be expressed in the form $u_i = f_i(\tau)$, τ being a function of $x_1, \cdots, x_m$. He studies the equations of isentropic flow in three dimensions, which constitute a system of this type, and obtains the general integral by the classical method. In this case the parameter τ is proportional to the wave velocity. The specialization to one dimension yields the familiar expression of Riemann for progressive waves of finite amplitude. *R. N. Goss* (San Diego, Calif.).

Freud, G. Über das Randwertproblem dritter Art der Potentialtheorie. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 223–238 (1955). (Hungarian. Russian and German summaries)

The author gives an explicit formula for the Green's function of the Laplace equation with respect to the third boundary value problem $T + dT/dn = 0$, in case of the half plane and half space. With its aid he solves the third boundary value problem of potential theory for these domains. *P. D. Lax* (New York, N.Y.).

Hartman, Philip; and Wintner, Aurel. Partial differential equations and a theorem of A. Kneser. Rend. Circ. Mat. Palermo (2) **4** (1955), 237–255.

This paper contains a proof of this result, among others: If f is a nonpositive Hölder continuous function in Euclidean 3-space outside the origin, then the equation $\Delta u + fu = 0$ has a unique positive solution u which behaves like $1/r$. The proof uses the following tools: the strong maximum principle, a comparison theorem, the minimum characterization of the lowest eigenvalue and the following principle: if w is a positive solution of $\Delta w + gw = 0$ over a domain, then the first eigenvalue of $\Delta w + gw$ over the domain with boundary condition $w = 0$ is positive. The asymptotic behaviour at infinity of the solution is investigated under various assumptions about f.

P. D. Lax (New York, N.Y.).

Nagumo, Mitio; and Anasako, Yukio. On Perron's method for the semi-linear hyperbolic system of partial differential equations in two independent variables. Osaka Math. J. **7** (1955), 179–184.

A simplified proof of the existence of a unique solution of a first order hyperbolic system of semilinear equations for functions of two independent variables, under the assumption that the coefficients and initial values have continuous first derivatives.

P. D. Lax.

Šatunov, M. P. Solution of a mixed problem for the wave equation. Leningrad. Gos. Univ. Uč. Zap. **144**. Ser. Mat. Nauk 23 (1952), 270–273. (Russian)

Define $U(X, t)$ by

$\Delta U - U_{tt} = F(X, t)$, $U|_{t=0} = U_t|_{t=0} = 0$, for $X \in D$, $U|_S = 0$,

where $X=(x, y)$, and S is the boundary of the region D in the x, y plane. The author proves existence of U under suitable hypotheses, notably that

$$(*) \quad \left|\frac{\partial F}{\partial x}\right| + \left|\frac{\partial F}{\partial y}\right| + |F| + \left|\frac{\partial F}{\partial t}\right| + \cdots + \left|\frac{\partial^7 F}{\partial t^7}\right| < Ae^{\alpha_0 t}$$

for some A, α_0, and also, when $t=0$,

$$(**) \quad F = \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial t^2} = \cdots = \frac{\partial^5 F}{\partial t^5} = 0.$$

The method involves taking the Laplace transform

$$v(X, \lambda) = \int_0^\infty e^{-\lambda t} U(X, t)\, dt$$

and noticing that, by integrating by parts and applying conditions (*) and (**), one gets (***) $\Delta v = \lambda^2 v + g$, $v|_s = 0$ with $|g(X, \lambda)| < B/|\lambda|^7$ for $\mathrm{Re}(\lambda) > \alpha_0$. Problem (***) can be reduced by a standard method (using Green's function G for the Dirichlet problem for D) to a Fredholm integral equation, to which Hilbert-Schmidt theory can be applied, yielding the result that $v = \sum C_k v_k(X)$, where $v_k(X)$ are the eigenfunctions of $v_k = -\lambda_k^2 \int_D G v_k dX$. The coefficients C_k can be determined essentially by substitution in (***). It remains to study the dependence of v on λ (which is easily done by considering separately each of several possible cases, such as $[\mathrm{Re}(\lambda)]^2 > [\mathrm{Im}(\lambda)]^2$, etc.), and to invert the Laplace transform by means of the Bromwich integral. *R. B. Davis* (Syracuse, N.Y.).

Sobolev, S. L. An instance of a correct boundary problem for the equations of string vibration with the conditions given all over the boundary. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 707–709. (Russian)

Let $u_1 = \partial U/\partial t$, $u_2 = \partial U/\partial x$. The system

$$(*) \quad \partial u_1/\partial t = \partial u_2/\partial x, \quad \partial u_2/\partial t = \partial u_1/\partial x$$

is equivalent to the equation of the vibrating string. Let the functions u_1, u_2 satisfy on the sides of the square $-1 \leqq x \leqq 1$, $-1 \leqq t \leqq 1$ the conditions

$[a_1(t)u_1 + b_1(t)u_2]_{x=-1} = f_1(t)$, $[a_3(x)u_1 + b_3(x)u_2]_{t=-1} = f_3(x)$

plus two similar conditions on the sides (**) $x=1$, $t=1$. Under broad assumptions on the functions a, b, the problem of solving (*) subject to (**) is correctly set. The general solution of (*) is $u_{1,2} = \phi_1(x+t) \pm \phi_2(x-t)$. Application of the boundary conditions leads to a system of four algebraic equations which are uniquely solvable if the determinant Δ is nonvanishing. When Δ has zeros of finite multiplicity at isolated points, one must impose additional conditions on $f_1, \cdots, f_4$. In the general case one can construct solutions using Green's formula. The author promises a forthcoming work on the correct formulation of boundary-value problems for differential equations of a general type, of which the present note treats merely a simple example. *R. N. Goss* (San Diego, Calif.).

Maslennikova, V. I. The solution of a mixed problem for the unsteady motion of a rotating viscous fluid and a study of the differential properties of this solution. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 697–700. (Russian)

The author considers an equation of the form

$$Lv = \frac{\partial v}{\partial t} - [v \times w] - \mu \Delta v + \mathrm{grad}\, p = F(x, t),$$

where w and F are vector functions defined on a cylindrical domain $Q = \Omega \times (0 \leqq t \leqq 1)$ in three-space, μ a positive constant, and seeks a solution (v, p), where p is a scalar function on Q while v is a vector function on Q with div $v=0$ and all the components of v zero for $t=0$ and on the boundary of Ω for all t. A corresponding generalized problem for weak solutions is defined, and theorems on the existence, uniqueness, and regularity in the interior of Q for weak solutions are stated. Indications are given of some parts of the proofs using a method of Višik [same Dokl. (N.S.) **97** (1954), 193–196; MR **17**, 859]. *F. Browder* (New Haven, Conn.).

Èĭdel'man, S. D. On certain properties of solutions of parabolic systems. Ukrain. Mat. Ž. **8** (1956), 191–207. (Russian)

Les résultats de cet article sont pour l'essentiel contenus dans un mémoire plus récent du même auteur [Mat. Sb. N.S. **38(80)** (1956), 51–92; MR **17**, 857]. Les méthodes de démonstration sont les mêmes. *J. L. Lions* (Nancy).

★ **Leray, Jean. Intégrales abéliennes et solutions élémentaires des équations hyperboliques.** Second colloque sur les équations aux dérivées partielles, Bruxelles, 1954, pp. 37–43. Georges Thone, Liège; Masson & Cie, Paris, 1955.

The author constructs elementary (fundamental) solutions of homogeneous partial differential operators with constant coefficients as abelian integrals of holomorphic differential forms over cycles of the characteristic set of the operator in complex projective space relative to the intersection of the characteristic set with hyperplanes. For hyperbolic operators and real arguments this elementary solution vanishes outside the convex cone of of space-like directions. *P. D. Lax* (New York, N.Y.).

Višik, M. I. The problem of Cauchy with operators as coefficients, the mixed boundary problem for systems of differential equations and an approximate method of their solution. Mat. Sb. N.S. **39(81)** (1956), 51–148. (Russian)

Développement de notes dans Dokl. Akad. Nauk SSSR (N.S.) **97** (1954), 193–196; **99** (1954), 189–192; **100** (1955), 409–412 [MR **17**, 859].

Soit $A(t)$, $B(t)$, $C(t)$ des opérateurs non continus dans un espace de Hilbert H; t est un paramètre $\geqq 0$ (le temps); on cherche une fonction $t \to u(t)$, $t \geqq 0$, à valeurs dans H, telle que l'on puisse donner un sens à l'expression suivante

$$Lu(t) = A(t)\frac{d^2}{dt^2}u(t) + B(t)\frac{d}{dt}u(t) + C(t)u(t).$$

que l'on ait $Lu(t) = h(t)$, fonction donnée, et qu'enfin $u(0)$ et $du(0)/dt$ prennent des valeurs données.

Le but essentiel de l'article est de donner des conditions portant sur $A(t)$, $B(t)$, $C(t)$ pour que le problème çi-dessus admette une solution unique (solution dans un sens convenable).

Des résultats de ce genre ont été donnés ou annoncés par T. Kato [J. Math. Soc. Japan **5** (1953), 208–234; MR **15**, 437], K. Yosida [Proc. Japan. Acad. **30** (1954), 19–23, 273–275; MR **16**, 370], O. A. Ladyženskaya; [Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 207–210; MR **17**, 161], et J. L. Lions [C. R. Acad. Sci. Paris **240** (1955), 390–392; MR **16**, 927] (il n'y a aucune relation d'inclusion entre les divers résultats de ces A.).

Les hypothèses relatives aux nombreux théorèmes sont trop longues pour être rapportées ici. Signalons toutefois que l'A. suppose essentiellement que l'on peut écrire $A(t)$ (et aussi $B(t)$ et $C(t)$) sous la forme: $A(t)=G^*\tilde{A}(t)G$, où G est un opérateur linéaire fermé à ensemble de définition dense dans H à valeurs dans un deuxième espace Hilbertien H_1, où G^* est l'adjoint de G, et où $\tilde{A}(t)$ est un opérateur linéaire continu de H_1 dans lui-même (et en outre positif). Exemple de cette situation: $A(t)$ est un opérateur différentiel en x ($x \in$ à un ouvert borné de R^n), à coefficients dépendant de x et t, elliptique (en x, à t fixé), les conditions aux limites étant celles de Dirichlet; G est ici une variante simple des opérateurs de type „gradient" précédemment utilisés par l'A. [Mat. Sb. N.S. **29(71)** (1951), 615–676; MR **14**, 174]. (Pour des conditions aux limites plus générales, cf. Lions, loc. cit., avec $A(t)=0$; les cas $A(t)\neq 0$ peuvent être également atteints.)

Les méthodes de démonstration utilisent soit la technique des opérateurs non continus dans un espace de Hilbert, et des majorations simples, soit une méthode dite de Galerkine qui consiste à introduire des solutions approchées (à l'aide de bases; on est ainsi ramené à des equations différentielles ordinaires en t) et en déduire la solution par passage à la limite. *J. L. Lions* (Nancy).

Pettineo, Benedetto. Trattazione funzionale dei problemi al contorno relativi alle equazioni ed ai sistemi di equazioni lineari a derivate parziali. Rend. Circ. Mat. Palermo (2) **5** (1956), 101–116.

Picone computed the solutions of boundary values problems for linear partial differential equations by using a Fischer-Riesz system of integral equations derived from the Green's identity [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2** (1947), 365–371, 485–492, 717–725; MR **9**, 145, 286, 287]. This paper is devoted to a proof that under suitable hypotheses the solution of the derived Fischer-Riesz system is a "generalised" solution, in a proper sense, of the corresponding boundary value problem. The case of second order elliptic partial differential equation is first considered. An extension is outlined for non-elliptic second order equations. *G. Fichera.*

Boley, Bruno A. A method for the construction of Green's functions. Quart. Appl. Math. **14** (1956), 249–257.

The usual iterative method

$$G_{i+1}(P, Q)=G_0(P, Q)-\int_C G_0(R, Q)[\partial G_i(P, R)/\partial n_R]ds_R$$

for finding the Green's function $G(P, Q)=\lim G_i(P, Q)$ of a partial differential equation for a domain bounded by a curve C, given an "approximation" G_0 satisfying the proper differential equation and possessing the right singularity at $P=Q$ but the wrong boundary values, is replaced by the iteration

$$G_{i+1}(P, Q)=G_i(P, Q)-\int_C G_i(R, Q)[\partial G_i(P, R)/\partial n_R]ds_R.$$

For the special case of the Helmholtz equation $u_{xx}+u_{yy}=q(x, y)u(x, y)$, $q\geq 0$, this latter iteration is shown to converge, though not necessarily to $G(P, Q)$. Both methods are carried through in detail for a simple one-dimensional heat conduction problem; there the latter method turns out to converge (to the right value) somewhat faster.

H. C. Kranzer (New York, N.Y.).

Moses, H. E. Calculation of the scattering potential from reflection coefficients. Phys. Rev. (2) **102** (1956), 559–567.

The procedure of Jost and Kohn [Phys. Rev. (2) **87** (1952), 977–992; MR **14**, 556] is generalized and applied to one- and three-dimensional problems where V, the scattering potential, is not necessarily symmetric. In the one dimensional case the results are shown to conform with those obtained by the Gel'fand-Levitan procedure [see Kay, Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-74** (1955); Kay and Moses, ibid. No. **CX-20** (1955); Nuovo Cimento (10) **3** (1956), 276–304; MR **16**, 1113; **17**, 489, 971]. *N. Levinson.*

Brownell, F. H. An extension of Weyl's asymptotic law for eigenvalues. Pacific J. Math. **5** (1955), 483–499.

The author investigates the number of eigenvalues, smaller than t, $N(t)$ of the Laplace operator over a plane domain with zero boundary values. He conjectures that

$$N(t)=\frac{A}{4\pi}t-\frac{C}{4\pi}t^{1/2}+O(\ln t),$$

where A is the area, C the circumference of the underlying domain. This would be a refinement of the classical results of Weyl and Courant. With the aid of a recent estimate of Pleijel's [Medd. Lunds. Univ. Mat. Sem. Tome Supplémentaire (1952), 173–180; MR **14**, 561] valid for smooth domains:

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n(x_n+\omega)}=\frac{A}{4\pi}\frac{\ln\Delta}{\omega}+\frac{c}{\omega}+\frac{l}{8}\frac{1}{\omega^{3/2}}+O\left(\frac{1}{\omega^2}\right),$$

where λ_n is the nth eigenvalue, the author shows the validity of his result in the sense of a kind of Gaussian summability. He also verifies the result in case of a square, correcting thereby an error of Minakshisundaram [Proc. Symposium on Spectral Theory and Differential Problems, Oklahoma Agric. and Mech. Coll., 1951, pp. 325–332; MR **13**, 241]. *P. D. Lax* (New York, N.Y.).

See also: Mergelyan, p. 202; Gurevič, p. 216; Okubo, p. 231; Teleman, p. 232; Ghermănescu, p. 237; Pólya, p. 250; Hodge and Romano, p. 250; Schmieden und Müller, p. 252; Volt, p. 256; Rubinow and Wu, p. 257; Thompson, p. 264.

Difference Equations, Functional Equations

Gurevič, B. L. New types of fundamental and generalized function spaces and the Cauchy problem for systems of difference equations involving differential operations. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 1001–1003. (Russian)

L'A. utilise les résultats de L. Hörmander [C. R. Acad. Sci. Paris **240** (1955), 392–395; MR **16**, 720] et les siens [Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 893–895; MR **16**, 720] d'ailleurs moins généraux, résultats qui donnent de vastes classes d'espaces fonctionnels transformés isomorphiquement les uns dans les autres par transformation de Fourier; par transposition on définit la transformation de Fourier sur des espaces de fonctions analytiques. L'A. utilise ces notions pour l'étude de systèmes d'évolution de la forme

$$\frac{\partial}{\partial t}u(x, t)=p((2\pi i)^{-1}\partial/\partial x, T_h, t)u(x, t),$$

où p est une matrice, $T_h v(x)=v(x-h)$, $x \in R^n$. Tout ceci poursuit les travaux de Schwartz, Gel'fand et Šilov, Šilov, Kostyučenko [voir la bibliographie dans la dernière analyse citée ci-dessus, par exemple]. *J. L. Lions.*

Kurepa, Svetozar. On some functional equations. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. 11 (1956), 3–5. (Serbo-Croatian summary)

The functional equations

(1) $f(x_1+x_2, x_3)+f(x_1, x_2)=f(x_2, x_3)+f(x_1, x_2+x_3)$,

(2) $f(x_1+x_2, x_3, x_4)+f(x_1, x_2, x_3+x_4)=$
$$f(x_2, x_3, x_4)+f(x_1, x_2+x_3, x_4)+f(x_1, x_2, x_3),$$

(3) $f(x_1+x_2, x_3, x_4, x_5)+f(x_1, x_2, x_3+x_4, x_5)$
$$+f(x_1, x_2, x_3, x_4)=f(x_2, x_3, x_4, x_5)$$
$$+f(x_1, x_2+x_3, x_4, x_5)+f(x_1, x_2, x_3, x_4+x_5)$$

are satisfied by the functions

(I) $f(x_1, x_2)=[G(x_1+x_2)-G(x_2)-G(x_1)+a$,

(II) $f(x_1, x_2, x_3)=$
$$[G(x_1+x_2, x_3)+G(x_1, x_2)-G(x_1, x_2+x_3)-G(x_2, x_3)]$$
$$+[H(x_1+x_2)-H(x_2)]+ax_1,$$

(III) $f(x_1, x_2, x_3, x_4)=[G(x_1+x_2, x_3, x_4)$
$+G(x_1, x_2, x_3+x_4)-G(x_2, x_3, x_4)-G(x_1, x_2+x_3, x_4)$
$-G(x_1, x_2, x_3)]+[H(x_1+x_2, x_3)-H(x_1, x_2+x_3)$
$-H(x_2, x_3)]+[K(x_1+x_2)-K(x_2)-K(x_1)]+a$,

respectively, where a is an arbitrary constant and G, H, K are arbitrary functions. If f is a differentiable function satisfying the equation (e) $(e=1, 2, 3)$, then it has the form (E) $(E=\mathrm{I, II, III})$ with a as an arbitrary constant and G, H, K as arbitrary differentiable functions. The proof for this theorem is explicitly given in the case of (2). *A. Rosenthal* (Lafayette, Ind.).

See also: Belluigi, p. 265.

Calculus of Variations

Kazimirov, V. I. On the semicontinuity of integrals of the calculus of variations. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 3(69), 125–129. (Russian)

Let $X=(x_1, \cdots, x_m)$, $U=(u_1, \cdots, u_l)$, $P=(p_1, \cdots, p_k)$ denote elements of finite dimensional space, Ω a bounded region in X space, and $F(X, U, P)$ continuous together with its first partial derivatives in the variables p_i. The author proves the following general semicontinuity result. Theorem. Let F be non-negative and positive semiregular with respect to P. Let $U_n(X)$, $P_n(X)$ be any sequences in $L_\alpha(\Omega)$, $\alpha>1$, such that $U_n(X)$ tends to a limit $U_0(X)$ strongly in $L_\alpha(\Omega)$ and $P_n(X)$ tends to a limit $P_0(X)$ weakly in $L_\alpha(\Omega)$. Then

$$\liminf_{n\to\infty} \int_\Omega F[X, U_n(X), P_n(X)]d\Omega \geq \int_\Omega F[X, U_0(X), P_0(X)]d\Omega.$$

For the special case where $P_n(X)$ is the set of partial derivatives $\partial u_n^i/\partial x^j$ for $n=0, 1, 2, \cdots$ this a consequence of a result of Morrey [Univ. California Publ. Math. 1 (1943), 1–130, p. 34; MR 6, 180]. *W. H. Fleming.*

See also: Berger, p. 247; Piątek, p. 250; Freiberger and Tekinalp, p. 250; Rayski, p. 260; Markowitz, p. 267.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Dieudonné, Jean. Sur les générateurs des groupes classiques. Summa Brasil. Math. 3 (1955), 149–179.

The study of the structure of the classical linear groups is based on the fact that they are generated by certain elementary transformations which leave a hyperplane element-wise invariant [see Dieudonné, La géométrie des groupes classiques, Springer, Berlin, 1955; MR 17, 236]. It is thus natural to ask for the minimum number of these transformations needed to represent a given element of a given group. The problem was solved by Scherk [Proc. Amer. Math. Soc. 1 (1950), 481–491; MR 12, 157] for the orthogonal group over a field not of characteristic 2. The author in this paper treats the problem for the groups $GL_n(K)$, $Sp_{2m}(K)$ and $U_n(K, f)$, the last of which includes the orthogonal group. If u is an element different from the identity element e in one of these groups, let $w=u-e$ and denote the rank of w by r. Then, with certain exceptions, u can be written as a product of at most r of the elementary transformations in the given group. In the exceptional cases, r must be replaced by $r+1$ for $GL_n(K)$ and $Sp_{2m}(K)$ and by $r+2$ for $U_n(K, f)$. The description of the exceptional transformations, which is too long to be given here for all the different cases, varies with the group in question and, as might be expected, the case of characteristic 2 and certain finite fields require special treatment. We mention here only one result for each of the three types of groups considered. Let E be the vector space on which the groups operate and set $W=w^{-1}(0)$. Denote by $\bar{u}$ the transformation induced on E/W by u.

The only elements u in $GL_n(K)$ which are not products of $r-1$ transvections and one transvection or dilation are those for which $r>1$ and $\bar{u}$ is a homothetic transformation different from the identity. In this case u is a product of r transvections and one transvection or dilation.

If K is a field with more than two elements, then the only elements u of $Sp_{2m}(K)$ which are not products of r symplectic transvections are those for which $(x, u(x))=0$ for every $x \in E$. In this case u is a product of $r+1$ symplectic transvections.

If K is non-commutative then the only elements u in $U_n(K, f)$ which are not products of r quasi-symmetries are those for which $w(x)$ is isotropic for every $x \in E$. In this case u is a product of $r+2$ but not $r+1$ quasi-symmetries. This also holds when K is commutative except when the involution in K is different from the identity and the subfield of self-adjoint elements contains only two elements. *C. E. Rickart* (New Haven, Conn.).

Hewitt, Edwin; and Zuckerman, Herbert S. Harmonic analysis for certain semigroups. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 253–255.

Let G be a commutative, not necessarily finite, semi-

group, and let $L_1(G)$ be the totality of complex-valued functions f on G metrized by the norm $\|f\|=\sum_{u\in G}|f(u)|$. $L_1(G)$ is a Banach algebra by the convolution $f*g=\sum_{x=uv}f(u)g(v)$. A number of results are announced. For example, (i) the general homomorphism of $L_1(G)$ onto the complex number field has the form

$$f\to\sum_{u\in G}f(u)\chi(u),$$

where $\chi(u)$ is a semi-character of G, viz. a bounded complex function χ on G satisfying $\chi(u)\chi(v)=\chi(uv)$, (ii) the Gel'fand-Jacobson radical of $L_1(G)$ consists exactly of those $f\in L_1(G)$ for which $\sum_{y\in T_x}f(y)=0$ for all

$T_x=$
$\{y: y\in G, xy^n=y^{n+1}, yx^n=x^{n+1}$ for some positive integer $n\}$,

(iii) the algebra $L_1(G)$ has a unit if and only if G contains a finite subset A such that, for every $x\in G$, there exists $a\in A$ such that $ax=x$. *K. Yosida* (Tokyo).

See also: Mori, p. 187; Shephard, p. 191; Kneser, p. 192; Jerison and Rabson, p. 197; Matsushita, p. 203; Weston, p. 203; Kuiper, p. 232.

Lie Groups and Algebras

Sul'din, A. V. On certain subgroups of simple real Lie groups. Uč. Zap. Kazan. Univ. **115** (1955), 157–158. (Russian)

Let G_k be a complex simple Lie group and G_b a real form of G_k. Let K be the subgroup of G_k generated by the positive root vectors and consider also all subgroups conjugate to K. The first problem is to describe the intersections of these subgroups with G_b (up to conjugacy in G_b). The second problem is to identify the transitivity classes (orbits) of the homogeneous space G_k/K under the transformations of G_b. Gel'fand and Graev solved these problems when G_k is the complex and G_b the real unimodular group [Dokl. Akad. Nauk SSSR (N.S.) **86** (1952), 461–463; MR **14**, 448]. The author considers the case where G_k is the complex unimodular group and G_b the subgroup leaving invariant the Hermitian form

$$\phi(x)=x_1\bar{x}_1+\cdots+x_{n+1-d}\bar{x}_{n+1-d}-x_{n+2-d}\bar{x}_{n+2-d}-\cdots-x_{n+1}\bar{x}_{n+1}.$$

It is shown that in problem one only a certain special finite set of conjugates of K need be considered, and that in problem two there are exactly as many orbits as there are symmetric matrices permuting the axes and having exactly d characteristic numbers equal to -1.
A. Shields (Ann Arbor, Mich.).

Simoniya, V. T. Representation of simple Lie algebras with two-dimensional maximal solvable subalgebras. Soobšč. Akad. Nauk Gruzin. SSR **17** (1956), 393–400. (Russian)

Formulas with rational coefficients are given for irreducible representations as described in the title.
R. A. Good (College Park, Md.).

Harish-Chandra. Invariant differential operators on a semisimple Lie algebra. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 252–253.

Let E_0 be a finite dimensional vector space over the field R of real numbers and E the complexification of E_0. For any X in E_0 let $\partial(X)$ be the differentiation in the direction of X. Then ∂ can be extended uniquely to an isomorphism of the symmetric algebra $S(E)$ into the algebra of differential operators on E_0. Let g_0 be a real semi-simple Lie algebra and h_0 a Cartan subalgebra of g_0; let g, h be their complexifications. The author has shown previously [Trans. Amer. Math. Soc. **75** (1953), 185–243, p. 194; MR **15**, 100] that one can identify $S(g)$ with the algebra $Q(g)$ of polynomial functions on g, by means of the fundamental bilinear form of g. Let $I(g)$ be the subalgebra of $Q(g)$ consisting of the polynomial functions which are invariant under the adjoint group. Let $I(h)$ denote the subalgebra of those elements of $S(h)$ which are invariant under the Weyl group W (of g with respect to h). For any $p\in I(g)$ let $\bar{p}$ denote the restriction of the polynomial function p to h. Let Π denote the product of all positive roots (of g with respect to h under some fixed order) as an element of $S(h)$. Theorem 1 asserts the following. Let f be an indefinitely differentiable function on g_0 invariant under the adjoint group. Then for any $p\in I(g)$ and any $H\in h_0$ one has $\Pi(H)\partial(p)f(H)=\partial(\bar{p})g(H)$, where g is the function on h_0 given by $g(H)=\Pi(H)f(H)$. From this the author obtains a continuous map from the space of indefinitely differentiable functions on g_0 with bounded derivatives into the corresponding space of functions defined on the subset h_0' of those elements H of h_0, where $\Pi(H)\neq0$. No proofs are given. *F. I. Mautner.*

Harish-Chandra. A formula for semisimple Lie groups. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 538–540.

This is a continuation of the paper reviewed above. Using the same notation as above, theorem 1 asserts that for suitably restricted functions f on g_0,

$$\lim_{H\to0}\partial(\Pi)F_f(H)=cf(0),$$

where H lies in a connected component of h_0' and $F_f(H)$ is obtained from f by an explicit formula involving an integral of f over a certain coset space. Moreover, c is a real constant which equals zero if h_0 does not satisfy a certain maximality condition. Let h_0 satisfy this condition, and $h_1, h_2, \cdots, h_r$ be all the distinct connected components of h_0' and c_j the real constant in theorem 1 corresponding to h_j. Theorem 2 asserts that $c_1+c_2+\cdots+c_r\neq0$. These results are now applied to obtain an expression for the value of (suitably restricted) functions on the group G at the identity. This formula is an essential step in the derivation of the explicit Plancherel formula for those semi-simple Lie groups for which it is known. It may therefore turn out to be of importance in the as yet unsolved problem of obtaining the explicit Plancherel formula of an arbitrary (real) semi-simple Lie group.
F. I. Mautner (Baltimore, Md.).

See also: Kostrikin, p. 188.

Topological Vector Spaces

Köthe, Gottfried. Bericht über neuere Entwicklungen in der Theorie der topologischen Vektorräume. Jber. Deutsch. Math. Verein. **59** (1956), Abt. 1, 19–36.

Expository paper covering pretty much the same ground as the reviewer's report [Bull. Amer. Math. Soc. **59** (1953), 495–512; MR **15**, 963]. More details are given concerning the historical evolution of the theory, as well as on some recent results of Grothendieck on the space L^p. There is also an interesting description of a definition of the

space of distributions on a compact interval as inductive limit of Banach spaces, after E. König and J. Sebastião e Silva. *J. Dieudonné* (Evanston, Ill.).

Cotlar, M. A combinatorial inequality and its applications to L^2-spaces. Rev. Mat. Cuyana 1 (1955), 41–55 (1956).

Let $k \in L_1(R^n)$ $(n=1, 2, 3, \cdots)$, and let $*$ denote convolution. Suppose that $k=k_1+k_2+\cdots+k_N$, where $\|k_j*k_l\|_1 \leq C\cdot 2^{-|j-l|}$ $(j, l=1, 2, \cdots, N)$ and $\|k_j\|_1 \leq C$. Then, for every $f \in L_2(R^n)$, the inequality

$$\|f*k\|_2 \leq 16C^2 \cdot \|f\|_2$$

holds. It is easy to prove from this the existence and boundedness of the Hilbert transform for $f \in L_2(R^1)$. Part of the theory of 2-dimensional Hilbert transforms due to Calderón and Zygmund [Acta Math. 88 (1952), 85–139; MR 14, 637] can also be reconstructed. The proof of the main theorem is elementary, depending upon an ingenious combinatorial lemma. *E. Hewitt* (Seattle, Wash.).

Choquet, Gustave. Existence des représentations intégrales au moyen des points extrémaux dans les cônes convexes. C. R. Acad. Sci. Paris 243 (1956), 699–702.

Let X be a locally convex topological vector space, $C\subset X$ a convex cone, $V\subset X$ an affine variety which meets every generator of C (not at the vertex of C), and suppose $B=C\cap V$ is compact. B is called the base of C. Let $E\subset B$ be the set of extreme points of B. For every positive Radon measure μ on B, the resultant of μ is defined as $x_\mu=\int_B x d\mu$. The main result of this paper is that if B is also metrizable then, for every $x \in C$, there is a measure μ, whose support is E, such that $x=x_\mu$. *C. Goffman.*

Harrop, R.; and Weston, J. D. An intersection property in locally convex spaces. Proc. Amer. Math. Soc. 7 (1956), 535–538.

In a Banach space it is known that the intersection of a nesting sequence of closed spheres is non-empty. The present paper generalizes this result both as to space and the type of set. Let X be a locally convex linear Hausdorff space and suppose that B is a sequentially closed bounded set whose closed symmetric convex hull is likewise sequentially complete. Suppose that $\{\lambda\}$ is a directed set and the sets $B_\lambda=x_\lambda+\rho_\lambda B$, $\rho_\lambda\geq 0$, are such that $B_\lambda\subset B_\mu$ whenever $\lambda>\mu$. Then $x=\lim x_\lambda$ and $\rho=\lim \rho_\lambda$ exist and $\bigcap B_\lambda=x+\rho B$. The paper is quite readable.

R. S. Phillips (Los Angeles, Calif.).

Pinsker, A. G. On representation of a K-space as a ring of self-adjoint operators. Dokl. Akad. Nauk SSSR (N.S.) 106 (1956), 195–198. (Russian)

For a set S of bounded self-adjoint operators on a Hilbert space H, let S' be the set of all bounded self-adjoint operators commuting with every operator in S. Suppose that S is a set of bounded, self-adjoint, mutually commuting operators. Then the ring S'', under the usual ordering for self-adjoint operators, is a partially ordered vector space. Actually S'' is a complete vector lattice, or K-space in the author's terminology (every bounded set admits a least upper bound). The author cites an unpublished 1954 dissertation of L. V. Lyubovin for this fact. However, it is a simple consequence of v. Neumann's general theory of rings S'' [Math. Ann. 102 (1929), 370–427] and Stone's representation theory for such rings [Proc. Nat. Acad. Sci. U.S.A. 26 (1940), 280–283; MR 1, 338]. Note that the partially ordered set of all self-adjoint bounded operators is not even a lattice. The present note characterizes intrinsically the K-spaces X that are algebraically and order-wise isomorphic to rings S''. Given a transfinite sequence $\{x_\alpha\}$ of elements of X, one writes $x_\alpha\to x$ if

$$\inf_\alpha [\sup_{\beta>\alpha}(x_\beta)]=\sup_\alpha [\inf_{\beta>\alpha}(x_\beta)]=x.$$

A linear functional f on X is said to be completely additive if $x_\alpha\to x$ implies $f(x_\alpha)\to f(x)$. Then X is isomorphic to some ring S'' if and only if for every $x\neq 0$ in X, there is a completely additive functional f such that $f(x)\neq 0$.

E. Hewitt (Seattle, Wash.).

Pták, Vlastimil. Concerning spaces of continuous functions. Czechoslovak Math. J. 5(80) (1955), 412–431. (Russian summary)

Let T be a completely regular topological space, and let $C(T)$ be the linear space of all real-valued continuous functions on T. Let hT be the unique Q-space such that $T\subset hT$, $T^-=hT$, and every function in $C(T)$ admits a continuous extension over hT. [For the construction and elementary properties of hT, see Hewitt, Trans. Amer. Math. Soc. 64 (1948), 45–99; MR 10, 126.] Thus $C(T)$ and $C(hT)$ are algebraically and order-wise isomorphic. Now topologize $C(T)$ by the pseudo-norms $\|f\|_K=\max|f(t)|:t\in K$, where K runs through all compact subsets of T. Let $M(T)$ be the space of all linear functionals on $C(T)$ that are continuous in this K-topology. It is clear that $M(T)\subset M(hT)$, and that the inclusion may be proper. The theme of this paper is that a functional $r \in M(hT)$ still has some sort of continuity that enables one to describe the elements of $M(hT)$ in terms of $C(T)$ alone. {Reviewer's note. Plainly some such characterization must exist, since hT, and hence its compact subsets, can be identified by the algebraic structure of $C(T)$.} The p-topology of $C(T)$ is the topology of $C(T)$ regarded as a subset of the Cartesian product R^T, R being the real numbers. The weak topology of $C(T)$ is the usual weak topology defined in terms of the elements of $M(T)$. The first main theorem is the following. Let r be a linear functional on $C(T)$. Then the following conditions are equivalent. (1) $r \in M(hT)$. (2) r is p-continuous on every symmetrical convex and p-compact subset of $C(T)$ that contains a countable dense subset. (3) r is weakly continuous on every symmetrical convex and p-compact subset of $C(T)$ that contains a countable dense subset. (4) r is p-continuous on every symmetrical convex and weakly compact subset of $C(T)$ that contains a countable dense subset. (5) r is weakly continuous on every symmetrical convex and weakly compact subset that contains a countable dense subset. A second, analogous theorem is also proved. The proof is based on an earlier paper of the author's [Czechoslovak Math. J. 4(79) (1954), 175–186; MR 16, 595].

The author also studies hT as a subset of the Čech-Stone compactification βT. He calls a point $s \in \beta T$ a Hewitt point if every $f \in C(T)$ admits a continuous extension over $T\cup\{s\}$. Then hT, the Hewitt closure of T, is the set of all Hewitt points of βT. A similar characterization of hT has been given by Gillman, Henriksen, and Jerison [Proc. Amer. Math. Soc. 5 (1954), 447–455, p. 450; MR 16, 607]. The author also gives a simple and interesting example of a completely regular space that is not countably compact and in which all continuous real-valued functions are bounded.

E. Hewitt.

Schwartz, Laurent. Espaces de fonctions différentiables à valeurs vectorielles. J. Analyse Math. 4 (1954/55), 88–148.

Dans ce travail l'auteur démontre en détail quelques-uns de ses résultats exposés dans son Séminaire de 1953–54 [MR 17, 764]. *J. Sebastião e Silva* (Lisbonne).

Brodskiĭ, M. S. Characteristic matrix functions of linear operators. Mat. Sb. N.S. 39(81) (1956), 179–200. (Russian)

A is an operator with imaginary part $((A-A^*)/i)$ completely continuous; $\{e_\alpha\}$ is an orthonormal basis of eigenvectors of $(A-A^*)/i$ in $\mathcal{E}$, where $\mathcal{E}$ is a linear set in the closure of the range of $(A-A^*)/i$; $\{w_\alpha\}$ is the set of corresponding eigenvalues. Let Ω be the matrix diag (w_α) and J and Π any two matrices such that $\Pi^*J\Pi=\Omega$, $J=(\pm\delta_{ij})$, Π completely continuous such that $\Pi\xi=0$ implies $\xi=0$. Then

$$W(\lambda)=I-i\Pi\|((A-E)^{-1}e_\alpha, e_\beta)\|\Pi^*J,$$

E the projection on $\mathcal{E}$, is the (ambiguously defined) characteristic of A. $W(\lambda)$ has these properties: (1) Analyticity in a region G which results from the deletion of a bounded set of real and a bounded set of complex points which have no point of condensation; (2) $W(\lambda)=I+TJ\lambda^{-1}+\cdots$, where T is completely continuous; (3) complete continuity of $W(\lambda)-I$ in G; (4) $W(\lambda)JW^*(\lambda)\geqq J$ in $G\cap(\Im\lambda\geqq 0)$ with equality if and only if $\Im\lambda=0$. Theorem 1: If $W(\lambda)$ has properties (1)–(4), there is an operator A of which $W(\lambda)$ is the characteristic.

Let H_E be the subspace generated by $A^{*n}e_\alpha$ $(n=0, 1, 2, \cdots; \alpha=1, 2, \cdots)$, and A_E the restriction of A to H_E. Theorem 2: If $A^{(1)}$ and $A^{(2)}$ are defined in $H^{(1)}$ and $H^{(2)}$ respectively, and if they have characteristics $W^{(1)}(\lambda)$ and $W^{(2)}(\lambda)$ respectively which are equal; then $A_E^{(1)}(1)$ and $A_E^{(2)}(2)$ are unitarily equivalent. Theorem 3: If $H_E=H$, then the spectrum of A is the set of singularities of $W(\lambda)$.

If P_0 is a projection, let $A_0=P_0A$, $E_0=P_0\mathcal{E}$, $H_0=P_0H$, let $e_\alpha^{(0)}$, $w_\alpha^{(0)}$ have obvious meanings. Then

$$W_0(\lambda)=I-i\Pi_0\|(A_0-E)^{-1}e_\alpha^{(0)}, e_\beta^{(0)}\|\Pi_0^*J,$$

where $\Pi_0=\Pi U_0$, $U_0=\|(e_\alpha, e_\beta^{(0)})\|$; $\Omega=\mathrm{diag}\,(w_\alpha^{(0)})$, is called the projected characteristic of A on H_0, $\mathrm{Pr}_{H_0}W(\lambda)$. Theorem 4: If H_1 reduces A and $H_2=H_1^\perp$, then $W(\lambda)=W_2(\lambda)W_1(\lambda)$, where $W_k(\lambda)$ is the projected characteristic of A on H_k. Theorem 5: If $H_1\subset H_2\subset H$, then

$$\mathrm{Pr}_{H_1}(\mathrm{Pr}_{H_2}W(\lambda))=\mathrm{Pr}_{H_1}W(\lambda).$$

Theorem 6: If $H=H_1\oplus\cdots\oplus H_n$, $\tilde{H}_k=H_1\oplus\cdots\oplus H_k$ $(k\leqq n-1)$, where $\tilde{H}_k$ reduces A, then

$$W(\lambda)=\mathrm{Pr}_{H_n}W(\lambda)\mathrm{Pr}_{H_{n-1}}W(\lambda)\cdots\mathrm{Pr}_{H_1}W(\lambda).$$

Detailed specialized applications of the above are also derived. *B. Gelbaum* (Minneapolis, Minn.).

Evgrafov, M. A. Completeness of the system of eigenfunctions of a certain class of operators in a linear topological space with a non-denumerable basis. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 13–15. (Russian)

In a continuation of earlier work [same Dokl. (N.S.) 107 (1956), 199–201; MR 17, 1111] the author states without proof the following results. (Notation and terminology as in the cited references.) In $\mathfrak{A}(\sigma(r))$ let A be an operator of the form:

$$A(F(x))=\rho(x)F(x)+\int_0^x \varepsilon_\lambda(x)F(\lambda)\,d\lambda.$$

Here $\varepsilon_\lambda(x)$ is some suitably restricted kernel (for details see the paper). A^* is defined by:

$$A^*(F(x))=\rho(x)F(x)+\int_x^\infty \varepsilon_x(\lambda)(F\lambda)\,d\lambda.$$

Then for each $\mu>0$ there is a unique function $\varphi_\mu(x)$ and a unique function $\psi_\mu(x)$ such that $A(\varphi_\mu(x))=\rho(\mu)\varphi_\mu(x)$ and $A^*(\psi_\mu(x))=\rho(\mu)\psi_\mu(x)$. Furthermore: $\varphi_\mu(x)=\delta(x-\mu)+a_\mu(x)$, $a_\mu(x)=0$ if $x<\mu$, and $\psi_\mu(x)=\delta(x-\mu)+b_\mu(x)$, $b_\mu(x)=0$ if $x>\mu$; $\int_0^\infty \varphi_\mu(x)\psi_\nu(x)dx=\delta(\mu-\nu)$. Conditions on the $a_\mu(x)$ and $b_\mu(x)$ are given which insure that the $\varphi_\mu(x)$ constitute a basis for $\mathfrak{A}(\sigma(r))$. *B. Gelbaum* (Minneapolis, Minn.).

Birman, M. Š. On the theory of self-adjoint extensions of positive definite operators. Mat. Sb. N.S. 38(80) (1956), 431–450. (Russian)

A well-written and partly expository article on recent work dealing with the self-adjoint extensions of a symmetric linear operator with a positive lower bound. Let S be the operator, $S^*\supset S$ its adjoint, U the null-space of S^* and T the Friedrichs extension of S. The fundamental result due to Višik [Trudy Moskov. Mat. Obšč. 1 (1952), 187–246; MR 14, 473] is that $D(S)$ is the direct sum $D(S)\dotplus T^{-1}U\dotplus U$ and that $\tilde{S}\supset S$ is a self-adjoint extension of S if and only if $D(\tilde{S})=D(S)\dotplus(T^{-1}+B)D(B)\dotplus U_0$. Here U_0 is any closed subspace of U and B is any self-adjoint operator on the complementary subspace. Interest is concentrated to the case when S has infinite deficiency indices and $\tilde{S}$ is bounded from below. Let B^{-1} be the inverse of the restriction of B to the orthogonal complement (in $U\ominus U_0$) of its null-space and extend B^{-1} (without changing to a new symbol) from $R(B)$ to $D(B^{-1})=R(B)\dotplus U_0$ by putting it equal to zero on U_0. It turns out that $\tilde{S}\geqq a$ if and only if $(B^{-1}f, f)\geqq a(f, f)+a^2(R_af, f)$, $(R_a=(T-a)^{-1}$, for every f in $D(B^{-1})$ and that $\tilde{S}$ is bounded from below if and only if $D(\tilde{S})=D(S)\dotplus D(B^{-1})$. Further, $\tilde{S}$ has a finite negative spectrum if and only if B^{-1} has. The paper ends with a similar theory for symmetric extensions of S. *L. Gårding.*

Halberg, Charles J. A., Jr.; and Taylor, Angus E. On the spectra of linked operators. Pacific J. Math. 6 (1956), 283–290.

Let X, Y, Z be complex linear spaces $Z\subset X\cap Y$. If X, Y are Banach spaces X_1, Y_1 with respect to the norms n_1, n_2, Z is a Banach space Z_N under the norm

$$N(z)=\max[n_1(z), n_2(z)]$$

and if T_1, T_2 are bounded linear operators over X_1, Y_1 respectively, then T_1 and T_2 are said to be linked if $T_1z=T_2z\in Z_N$ for every $z\in Z_N$. If Z_N is dense in X_1, then T_1, T_2 are said to be linked densely relative to X_1. The purpose of this paper is to investigate the relationships between the spectra $\sigma(T_1)$, $\sigma(T_2)$ of linked operators. It is shown by example that linked and densely linked operators may have different spectra. The principal theorem states that if T_1, T_2 are linked densely relative to X_1 and if $R_\lambda(T_1)z=R_\lambda(T_2)z\in Z_N$ for every $z\in Z_N$ and for every $\lambda\in\rho(T_1)\cap\rho(T_2)$, where $\rho(T)$ is the resolvent set of T and $R_\lambda(T)$ the resolvent corresponding to the value λ, then if C is any component of $\sigma(T_1)$, $C\cap\sigma(T_2)$ is non-empty. Various corollaries and extensions of this theorem are proved which provide even closer relationships between $\sigma(T_1)$ and $\sigma(T_2)$ in particular cases. Applications are given where X_1 and X_2 are l_p spaces and T an infinite matrix defining a bounded linear operator over both spaces. *R. E. Fullerton* (College Park, Md.).

Getoor, R. K. On semi-groups of unbounded normal operators. Proc. Amer. Math. Soc. 7 (1956), 387–391.

The following theorem is proved: Let $[N_t; t>0]$ be a semi-group (i.e. $N_{t+s}=N_tN_s$) of normal operators on a Hilbert space. Let D_t be the domain of N_t (each D_t is dense) and let $D=\bigcap_{t>0} D_t$. It is supposed that N_tx is weakly continuous as a function of t ($t>0$) for each fixed $x \in D$. Then there exists a unique complex spectral resolution $K(\Lambda)$ whose support is contained in $\lambda_1 \geqq 0$, $\lambda=\lambda_1+i\lambda_2$, such that

$$N_t=\int_{\lambda_1\geqq 0} \lambda_1^t \exp(i\lambda_2 t)K(d\lambda) \quad (t>0).$$

This theorem generalizes a similar theorem on semi-groups of bounded normal operators due to Sz. Nagy [Math. Ann. 112 (1936), 286–296]. *R. S. Phillips.*

Civin, Paul; and Yood, Bertram. Invariant functionals. Pacific J. Math. 6 (1956), 231–237.

Suppose G is an abelian group of bounded linear operators on a normed linear space E, and G_1 is the convex hull of E. Yood showed earlier [Proc. Amer. Math. Soc. 2 (1951) 225–233; MR 12, 716] that if G is bounded and $\inf_{T\in G}\|T(x_0)\|>0$ for some $x_0 \in E$, then E admits a nontrivial bounded linear functional f invariant under G (i.e., $fT=f$ for all $f \in G$). In the present paper, the authors seek the same conclusion without assuming boundedness of G, and are led to a notion of stability, the point $y \in E$ being called "stable" provided there exists K such that for each $U \in G_1$,

$$\inf_{T\in G_1, \|T\|\leqq K} \|TU(y)\| \leqq K \inf_{T\in G_1} \|TU(y)\|.$$

Their principal result asserts that if E is complete and there is a nonempty open set $S\subset E$ such that both x and $T(x)-x$ are stable for all $x \in S$, $T \in G$, then an invariant functional exists. There are an extension to solvable G's and an application to the existence of positive invariant functionals. *V. L. Klee, Jr.* (Seattle, Wash.).

Samuel, Pierre. L'espace des idéaux d'un anneau local. Bol. Soc. Mat. Mexicana (2) 1 (1956), 10–12.

Let $\mathfrak{o}$ denote a local ring [cf. the author's Algèbre locale, Gauthier-Villars, Paris, 1953; MR 14, 1012], and let Φ denote the set of all ideals of $\mathfrak{o}$. The author defines a (separated) metrizable uniform structure on Φ such that for any ideals $\mathfrak{a}$, $\mathfrak{b}$ of $\mathfrak{o}$, the mappings $(\mathfrak{a}, \mathfrak{b})\to\mathfrak{ab}$, and $(\mathfrak{a}, \mathfrak{b})\to\mathfrak{a}+\mathfrak{b}$ are uniformly continuous. If $d(\mathfrak{a})$ denotes the dimension of the local ring $\mathfrak{o}/\mathfrak{a}$, then d is an upper semi-continuous mapping of Φ into the (discrete) space of integers, but will not ordinarily be continuous. Moreover if $\mathfrak{o}$ is a complete local ring, then Φ is a complete uniform space. *M. Henriksen* (Princeton, N.J.).

See also: Mori, p. 187; Volpato, p. 198; Lehmann, p. 213; Bopp, p. 259; Ito, p. 222.

Banach Spaces, Banach Algebras

Singer, Ivan. Propriétés de la surface de la sphère unitaire et applications à la résolution du problème de l'unicité du polynôme de meilleure approximation dans des espaces de Banach quelconques. Acad. R. P. Romîne. Stud. Cerc. Mat. 7 (1956), 95–145. (Romanian. Russian and French summaries)

L'auteur se propose d'étudier le problème suivant: Soit E un espace de Banach, $V\subset E$ un sous-espace de dimension m ($1\leqq m<\infty$) et pour tout $x \in E$ soit $V(x)$ l'ensemble des $y \in V$ tels que $\|x-y\|=\inf_{z\in V}\|x-z\|$. Trouver des conditions nécessaires et suffisantes (V étant fixé) pour que $V(x)$ contienne un seul élément.

Pour ce faire, dans la première partie de l'article, il donne quelques propositions concernant la caractérisation des parties convexes maximales de la sphère $\sigma=\{x|\ \|x\|=1\}$ de E à l'aide des points extrémaux de la sphère $\sigma'=\{x'|\ \|x'\|=1\}$ du dual E'. Il donne ensuite plusieurs propositions (§ 3) sur les moyennes au sens de A. A. Markoff [Mat. Sb. N.S. 4(46) (1938), 165–191] {qui sont inexactes}. Dans la seconde partie l'auteur cherche a résoudre le problème formulé plus haut et donne le théorème suivant, qu'il considère comme principal: Pour que $V(x)$ contienne, quel que soit $x \in E$, un seul élément, il faut et il suffit que

$$\bigcap_{\{j|\ 1\leqq j\leqq m, c_j\neq 0\}} \{x|\ f_j(x)=c_j/|c_j|,\ \|x\|=1,\ \|x+y\|=1\}=\emptyset$$

pour tout $y \in V$, $y\neq 0$, tout ensemble $\{f_1, \cdots, f_m\}$ de points extrémaux de σ', deux à deux non opposés et tout ensemble $\{c_1, \cdots, c_m\}$ de nombres réels tels que $\sum_{j=1}^m c_j=1$ et $\sum_{j=1}^m c_jf_j(z)=1$ pour $z \in V$. {Il est vrai que la condition est nécessaire. Mais la démonstration donnée par l'auteur pour montrer que la condition est suffisante est incorrecte; elle contient plusieures inéxactitudes et en particulier s'appuie sur les propositions du § 3.} *C. T. Ionescu Tulcea* (Bucureşti).

Weston, J. D. A note on integration in vector spaces. J. London Math. Soc. 31 (1956), 399–400.

If f is a continuous function from a locally compact space to a locally convex vector space F, then F has a subspace Y such that Y contains the range of f and such that Y (when suitably renormed) is equivalent to the adjoint of a Banach space. This remark is used to suggest a definition of (weak) integration alternative to the one employed by Bourbaki. *P. R. Halmos* (Chicago, Ill.).

Darbo, Gabriele. Sulla permanenza di certe proprietà in una trasformazione dipendente da un parametro — un criterio di invertibilità completa. Rend. Sem. Mat. Univ. Padova 25 (1956), 357–370.

Let Φ_λ be a one parameter family of mappings from a set Ξ to a Banach space Σ where the parameter λ ranges over a real interval J. The lacunary radius $r(\Phi)$ of a mapping $\Phi=\Xi\to\Sigma$ is defined as the upper bound of the radii of all spheres in Σ which do not intersect $\Phi(\Xi)$. The fundamental theorem of this paper states that if Φ_λ satisfies: (I_1) for each $x \in \Xi$, $\Phi_\lambda(x)$ is differentiable with respect to $\lambda \in J$,

$$(I_2) \quad \left\|\frac{d}{d\lambda}\Phi_\lambda(x_1)-\frac{d}{d\lambda}\Phi_\lambda(x_2)\right\|\leqq A(\lambda)k(\|\Phi_\lambda(x_1)-\Phi_\lambda(x_2)\|),$$

where x_1, x_2 are any two points of Ξ, $A(\lambda)$ is positive and integrable over J and $k(\xi)\geqq 0$ and increasing for $\xi\geqq 0$, $k(\xi)/\xi$ is decreasing and $\int_0^\epsilon d\xi/k(\xi)=\infty$ for each $\epsilon>0$, then one of the following statements hold: (1) $r(\Phi_\lambda)\equiv 0$ for $\lambda \in J$, (2) $0<r(\Phi_\lambda)<\infty$ for all $\lambda \in J$ or (3) $r(\Phi_\lambda)=\infty$ for all $\lambda \in J$. From this theorem various other permanence properties of the family are derived. The principal results are concerned with the existence of inverses for all Φ_λ in case Φ_λ is invertible for a single value of λ. Several results concerning the existence of inverses for operators over Banach and Hilbert spaces are derived by imbedding the given operator and the identity operator in a family Φ_λ which satisfies (I_1) and (I_2). There are several trouble-

some misprints in the paper. In the last line on page 363 read $\theta/(1-\theta)$ instead of $(1-\theta)/\theta$ and in formula (27) on page 368 read $\|(1-\lambda)\Delta\Phi_0-\lambda\Delta\Phi_1)\|$ instead of $\|(1-\lambda)\Phi_0-\lambda\Phi_1\|$.
R. E. Fullerton (College Park, Md.).

See also: Prékopa, p. 197; Alexiewicz and Orlicz, p. 205; Crum, p. 207; Rutman, p. 214; Hewitt and Zuckerman, p. 217; Cotlar, p. 219; Nečepurenko, p. 235; Block and Rosenbloom, p. 235.

Hilbert Space

Zuhovickiĭ, S. I.; and Stečkin, S. B. On the approximation of abstract functions with values in Hilbert space. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 385–388. (Russian)

This note is closely connected with an earlier paper by the first-named author [Mat. Sb. N.S. **37**(39) (1955), 3–20; MR **17**, 388]. Notation and terminology are as in the review cited. Let $f_1, \cdots, f_N$ be linearly independent and strongly continuous mappings of Q into H (H is assumed here to be separable). For complex numbers $\alpha_1, \cdots, \alpha_N$, let $p_\alpha=\sum_{k=1}^N \alpha_k f_k$. Let ϕ be a strongly continuous map of Q into H. Theorem 1. There exists a function p_{α_0} such that

$$\max_{q\in Q} \|p_{\alpha_0}(q)-\phi(q)\|=\inf\{\max_{q\in Q} \|p_\alpha(q)-\phi(q)\|: \text{all } \alpha_1, \cdots, \alpha_N\}.$$

Theorem 3. The function p_{α_0} is uniquely determined by ϕ, for all ϕ, if and only if p_α vanishes nowhere on Q for every $\{\alpha_1, \cdots, \alpha_N\}\neq\{0, 0, \cdots, 0\}$. Proofs are sketched.

E. Hewitt (Seattle, Wash.).

Kac, I. On Hilbert spaces generated by monotone Hermitian matrix-functions. Har'kov Gos. Univ. Uč. Zap. **34**=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **22** (1950), 95–113 (1951). (Russian)

M. G. Kreĭn [Akad. Nauk Ukrain. RSR. Zb. Prac' Inst. Mat. **1948**, no. 10, 83–106; MR **14**, 56] has introduced the notion of a matrix function

$$\Sigma(\lambda)=(\sigma_{ij}(\lambda))_{i,j=1}^p \ (-\infty<\lambda<+\infty),$$

such that (a) the form

$$\sum_{i,j=1}^p \sigma_{ij}(\lambda)\xi_i\bar{\xi}_j$$

is a real-valued, non-decreasing function of λ for all complex numbers $\xi_1, \cdots, \xi_p$ (i.e., $\Sigma(\lambda)-\Sigma(\lambda')$ is positive definite if $\lambda>\lambda'$) and (b) $\Sigma(0)=0$, $\Sigma(\lambda-0)=\Sigma(\lambda)$ for all λ. Let $v(\lambda)=\mathrm{Trace}(\Sigma(\lambda))$. It is easy to show that $v(\lambda)$ is a monotone increasing, left-continuous function of λ. Furthermore, all of the functions $\sigma_{ij}(\lambda)$ are absolutely continuous with respect to λ. Let the Radon-Nikodym derivative $d\sigma_{ij}/dv$ be written as δ_{ij} $(i, j=1, \cdots, p)$. Consider the space L_Σ^2 consisting of all sequences $\{f_1, \cdots, f_p\}$ of v-measurable functions on $]-\infty, +\infty[$ for which

$$(f, f)=\sum_{i,j=1}^p \int_{-\infty}^{\infty} f_i(\lambda)\overline{f_j(\lambda)}\delta_{ij}(\lambda)dv(\lambda)<\infty.$$

With the obvious pointwise linear operations and with the inner product

$$(f, g)=\sum_{i,j=1}^p \int_{-\infty}^{\infty} f_i(\lambda)\overline{g_j(\lambda)}\delta_{ij}(\lambda)dv(\lambda),$$

L_Σ^2 is a Hilbert space. The present paper gives a detailed construction of L_Σ^2 from the subspace of all $\{f_1, \cdots, f_p\}$ with continuous f_i. The author also discusses the Hermitian and possibly unbounded operator

$$B: \{f_1(\lambda), \cdots, f_p(\lambda)\}\to\{\lambda f_1(\lambda), \cdots, \lambda f_p(\lambda)\}.$$

B has deficiency index (0, 0) and hence admits a self-adjoint extension. Spectral properties of this extension are discussed.

E. Hewitt (Seattle, Wash.).

Itô, Takasi. On the uniformly bounded commutative group of linear transformations in the Hilbert space. J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1956), 68–70.

Assumptions: G is a set of linear transformations; G contains $AB=BA$ with A and B; for fixed $0<a\leq b$, every A in G satisfies $a\|f\|\leq\|Af\|\leq b\|f\|$ for every f in the Hilbert space.

Theorem: There exists $S=S^*$ with $a\leq S\leq b$ such that SAS^{-1} is isometric for every A in G. If some A has a right inverse then its $S^{-1}AS$ is necessarily unitary; this occurs for every A if G is a commutative group of uniformly bounded linear transformations. This generalizes the result of B.Sz. Nagy who proved this theorem for one parameter groups.

Proof: Use the weak topology for bounded linear transformations. Let K be the convex closure of the set of all A^*A (here A varies over G). Then $a^2\leq P\leq b^2$ for every P in K. Use the coordinate representation:

$$P=\{(Px, y); x, y \text{ varying over the Hilbert space}\}.$$

Tychonoff's theorem shows that K is a compact Hausdorff space. It is easily verified that for each A in G, K is mapped into itself continuously by the mapping $A(P)=A^*PA$.

Now, application of the Markhoff-Kakutani fixed point theorem shows that there is at least one P_0 in K satisfying $A^*P_0A=P_0$ for all A in G. Choose S positive definite with $S^2=P_0$. Then for every A in G, $A^*SSA=SS$, hence $(SAS^{-1})^*(SAS^{-1})=1$, which means that SAS^{-1} is isometric.

{The reviewer notes that more general theorems have been obtained by Dixmier [Acta Sci. Math. Szeged **12** (1950), Pars A, 213–227, Th. 6; MR **12**, 267] and Kadison [Amer. J. Math. **77** (1955), 600–620, Th. 1; MR **17**, 285] using other methods. This suggests the possibility of sharpening the fixed point theorem.}

I. Halperin.

Itô, Takasi. On the maximal spectrality. J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1956), 71–73.

Following H. Nakano, a spectrality is a function $E(A)$ defined for a countably additive family of subsets A of a fixed set Ω; the values of $E(A)$ are projection operators on a Hilbert space satisfying natural additivity conditions. Definitions are given for maximal and discrete spectralities. The author proves that for a separable Hilbert space a spectrality is maximal if and only if it is discrete.

I. Halperin (Kingston, Ont.).

See also: Višik, p. 215; Pinsker, p. 219; Getoor, p. 221; Taylor, p. 260.

TOPOLOGY

General Topology

Slowikowski, W.; and Zawadowski, W. A generalization of maximal ideals method of Stone and Gelfand. Fund. Math. 42 (1955), 215–231.

A semiring [Vandiver, Bull. Amer. Math. Soc. 40 (1934), 914–920] is a non-void set $\mathfrak{A}$ with two binary associative operations, $+$ and $\cdot$, for which the left and right distributive laws hold. It is assumed throughout this paper that both $+$ and $\cdot$ are commutative, that there is a zero 0 ($0+x=x$ for all x) and an identity 1 ($1\cdot x=x$ for all x), and that $0\neq 1$. An ideal of $\mathfrak{A}$ is a non-void proper subset I of $\mathfrak{A}$ such that $a+b \in$ I if $a, b \in I$ and $ax \in I$ if $a \in I$ and $x \in \mathfrak{A}$. $\mathfrak{A}$ is called a positive semiring if $(1+x)^{-1}$ exists for all $x \in \mathfrak{A}$. Let $\mathfrak{M}$ be the set of all maximal ideals of $\mathfrak{A}$. Let Ω be the set $\{x: x \in \mathfrak{A}, x^{-1}$ exists$\}$. For a positive semiring $\mathfrak{A}$, the radical Rad $\mathfrak{A}$ is defined as $\bigcap\{M: M \in \mathfrak{M}\}$.

A number of simple facts about semirings are first proved. The following are a fair sample. Theorem 4. $\mathfrak{A}$ is positive if and only if $x+y \in M$ implies $x, y \in M$ for all $M \in \mathfrak{M}$. Theorem 5. Let A be a positive semiring. Then Rad $A=\{0\}$ if and only if for every $x\neq 0$, there is $y \in \Omega$ such that $x+y \in \Omega$. Theorem 8. Let $\mathfrak{A}$ be positive. The following conditions are equivalent. (a) For $M, N \in \mathfrak{M}$ and $M\neq N$ there exist x, y such that $xy \in$ Rad $\mathfrak{A}$, $x \notin M$, and $u \notin N$. (b) For every $M \in \mathfrak{M}$ and $x \notin M$, there are a, b such that $ab \in$ Rad $\mathfrak{A}$, $b \notin M$, and $x+a \in \Omega$. (c) If $x+y \in \Omega$, then there are $u, v \in \mathfrak{A}$ such that $uv \in$ Rad $\mathfrak{A}$ and $(x+u)$, $(u+v) \in \Omega$.

The authors next topologize $\mathfrak{M}$ by the method of M. H. Stone [Trans. Amer. Math. Soc. 40 (1936), 37–111]. [See also Gel'fand and Šilov, Mat. Sb. N.S. 9(51) (1941), 25–39; MR 3, 52; and Jacobson, Proc. Nat. Acad. Sci. U.S.A. 31 (1945), 333–338; MR 7, 110.] A sub-basis for open sets in $\mathfrak{M}$ are the sets $\Gamma_x=\{M: M \in \mathfrak{M}, x \notin M\}$, for all $x \in \mathfrak{A}$. It is easy to show that $\mathfrak{M}$ is a compact T_1-space with this Γ-topology. Theorem 14. Let $\mathfrak{A}$ be positive and suppose that if $x, y \in A$ and $x\neq y$, there is an $M \in \mathfrak{M}$ such that exactly one of x, y is in M. Then $\mathfrak{A}$ is isomorphic with a semiring of sets ($\cup$ and $\cap$ correspond to $+$ and $\cdot$) forming an open basis for the compact T_1-space $\mathfrak{M}$. Theorem 17. Let $\mathfrak{L}$ be a distributive lattice with 0 and 1 such that for all $x, y \in \mathfrak{L}$ with $x+y=1$, there are $u, v \in \mathfrak{L}$ for which the equalities $uv=0$, $u+x=1$, $v+x=1$ hold. Then $\mathfrak{L}$ is isomorphic with an open basis of sets for some compact Hausdorff space. As applications, the authors prove the known theorems that a compact Hausdorff space X is determined by its lattice of open sets and also by the semiring of non-negative continuous real-valued functions on X.

It would be interesting to know the relation between the radical Rad $\mathfrak{A}$ and the Jacobson radical for A introduced by Bourne [ibid. 37 (1951), 163–170; MR 13, 7].

E. Hewitt (Seattle, Wash.).

Iséki, Kiyoshi; and Miyanaga, Yasue. Notes on topological spaces. III. On space maximal ideals of semiring. Proc. Japan Acad. 32 (1956), 325–328.

[For parts I–II see same Proc. 32 (1956), 27–28, 171–173; MR 17, 880, 1116.] Terminology and notation are as in the preceding review. For $x \in \mathfrak{A}$, let

$$\Delta_x=\{M: M \in \mathfrak{M}, x \in M\}.$$

Then the sets Δ_x ($x \in \mathfrak{A}$) can be taken as a sub-basis for open sets, yielding a topology for $\mathfrak{M}$ (the Δ-topology) that is in general different from the Γ-topology described in the preceding review. $\mathfrak{M}$ is a Hausdorff space under the Δ-topology. The authors re-prove several of the results of the paper reviewed above. Their new results are the following. Lemma 1. Let $\mathfrak{A}$ be a positive semiring. Then every Γ-open subset of $\mathfrak{M}$ contains as Δ-open subset. Theorem 5. Let $\mathfrak{A}$ be a positive semiring. Then the Γ- and the Δ-topologies coincide if and only if, for all $a \in \mathfrak{A}$, there is $ab \in \mathfrak{A}$ such that $\Gamma_a=\Delta_b$.

E. Hewitt.

Iséki, Kiyoshi. Notes on topological spaces. V. On structure spaces of semiring. Proc. Japan Acad. 32 (1956), 426–429.

Notation and terminology are as in the second preceding review. An ideal P of $\mathfrak{A}$ is said to be prime if $ab \in P$ implies $a \in P$ or $b \in P$. The set $\mathfrak{P}$ of all prime ideals of $\mathfrak{A}$ is given a topology like the Γ-topology for $\mathfrak{M}$: the sets $\gamma_x=\{P: P \in \mathfrak{P}, x \notin P\}$ are an open sub-basis for $\mathfrak{P}$. The author shows that $\mathfrak{P}$ is a compact T_0-space with this γ-topology. Let rad $A=\bigcap\{P: P \in \mathfrak{P}\}$. Theorem: $\mathfrak{M}$ is dense in $\mathfrak{P}$ if and only if Rad A = rad A. Two other simple results are obtained.

E. Hewitt (Seattle, Wash.).

Iséki, Kiyoshi; and Miyanaga, Yasue. Notes on topological spaces. IV. Function semiring on topological spaces. Proc. Japan Acad. 32 (1956), 392–395.

Let X be a completely regular topological space, let $C^*(X)$ be the ring of all real-valued, bounded, continuous functions on X, and $C^+(X)$ the semiring of all non-negative functions in $C^*(X)$. (Notation and terminology are as in the third preceding review.) The connection between ideals in $C^*(X)$ and filters of closed sets in X has been completely established by the reviewer [Trans. Amer. Math. Soc. 64 (1948), 45–99; MR 10, 126] and Gillman, Henriksen, and Jerison [Proc. Amer. Math. Soc. 5 (1954), 447–455; MR 16, 607]. In the present note, the authors consider similar properties for ideals in the semiring $C^+(X)$, where X need not be completely regular, but only Hausdorff. Let I be an ideal in $C^+(X)$. Let I^* be the family of closed subsets $A \subset X$ such that for all closed F disjoint from A, there is an $f \in I$ such that $\inf f(x)>0$ on F. For a filter J of closed subsets of X, let $J^0=\{f: f \in C^+(X)$, $f(x)<\varepsilon$ for all x in some set $F \in J\}$. Then I^* is a filter of closed subsets, and J^0 is an ideal of $C^+(X)$. If X is completely regular, then J^0 is a proper ideal. An ideal I is said to be closed if $I^{*0}=I$, and a filter J of closed subsets is said to be closed if $J^{0*}=J$. There is a one-to-one correspondence between closed ideals and closed filters. If X is completely regular and I is a maximal ideal, then I^* is an ultrafilter. If X is normal and J is an ultrafilter of closed sets, then J^0 is a maximal ideal. Finally, if I_1 and I_2 are ideals and $I_1\cap I_2=0$, then $I_1^{*0}\cap I_2^{*0}=0$.

E. Hewitt (Seattle, Wash.).

Sen Gupta, H. M.; and Basu Mazumdar, N. C. A note on certain plane sets of points. Bull. Calcutta Math. Soc. 47 (1955), 199–201.

The authors construct a plane set E of first category and of measure zero in the unit square $0\leq x\leq 1$, $0\leq y\leq 1$, having the property that any straight line which is not parallel to either of the axes and which has a point in common with the unit square has at least one point in common with E. E is secured as a union of a denumerable

sequence of non-dense perfect sets E_i lying in the unit square. It is shown that the distance set of E_i $(i=1, 2, \cdots)$ is the interval $[0, \sqrt{2}]$. *W. R. Utz.*

Gladkiĭ, A. V. Concerning a paper of D. E. Men'šov. Mat. Sb. N.S. 39(81) (1956), 379–384. (Russian)

A non-denumerable class of pair-wise disjoint, coplanar continua is constructed, such that the set of all singular points of the continua is of the second category in the plane. (A singular point of a continuum is one at which the continuum is not locally connected.) Each continuum constructed is a spiral about a unit segment, with an infinite number of revolutions in any neighborhood of the segment, together with the unit segment itself. *E. Mendelson* (Cambridge, Mass.).

Banaschewski, Bernhard. Überlagerungen von Erweiterungsräumen. Arch. Math. 7 (1956), 107–115.

In this paper are proved several propositions basic to another paper [Canad. J. Math. 8 (1956), 395–398; MR 17, 1229]; and the basic notion is that of connected trace filters (see the cited paper; connected means that if the union of a disjoint pair of open sets belongs to the filter, then so does at least one of them). C. Chevalley's notion of a simply connected (SC) space is involved in the following main results: (1) If F is a locally connected extension of an SC space E such that the trace filter of each point in $F-E$ is connected, then F is SC; (2) For a one-point (p_∞) locally connected extension F of E, F is SC precisely when the trace filter of p_∞ is connected. There is discussion of prime ends, of covering spaces, and it is shown that a slight modification of the latter concept leads to a modification of SC which the Čech-Stone extension of E (and some others) enjoys when E is SC. *R. Arens.*

Grabiel, Federico. Directed sets and generalized limits. Rev. Soc. Cubana Ci. Fís. Mat. 3 (1956), 139–148. (Spanish)

An elementary expository article. *R. Arens.*

Papić, Pavle. Sur la séparation des ensembles. Bull. Soc. Math. Phys. Serbie 6 (1954), 158–161. (Serbo-Croatian summary)

Let R be a T_1-space with a ramified open basis, as defined earlier by the author [Hrvatsko Prirod. Društvo. Glasnik Mat. Fiz. Astr. Ser. II. 8 (1953), 30–43; MR 14, 1001]. Theorem 1. In R, every pair of disjoint closed sets is contained in a pair of complementary open sets. Theorem 2. For a zero-dimensional metric space M, the following assertions are equivalent: (1) M has a ramified open basis; (2) every pair of disjoint closed sets in M are contained in complementary open sets; (3) M is homeomorphic to a totally ordered space. *E. Hewitt.*

Mardešić, Sibe; et Papić, Pavle. Sur les espaces dont toute transformation réelle continue est bornée. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 10 (1955), 225–232. (Serbo-Croatian summary)

A topological space X is said to be pseudo-compact [Hewitt, Trans. Amer. Math. Soc. 64 (1948), 45–99; MR 10, 126] if every continuous real-valued function on X is bounded. The authors introduce a new notion, as follows. A topological space X is feebly compact if every infinite family $\{V_\alpha\}$ of non-void, open, pairwise disjoint subsets of X admits at least one point of accumulation. Theorem 1. Let X be a completely regular space. Then X is pseudo-compact if and only if it is feebly compact. Theorem 2. Let X be regular. Then X is feebly compact if and only if for every countable open covering $\{V_n\}_{n=1}^{\infty}$ of X, there is a finite subfamily $\{V_n\}_{n=1}^{N}$ such that $\bigcup_{n=1}^{N} \bar{V}_n = X$. A topological space X is called feebly normal if every pair of disjoint closed sets one of which is countable and isolated can be separated by disjoint open sets. Theorem 4. Let X be feebly normal and a T_1-space. Then X is feebly compact if and only if X is countably compact. Several other similar results are also given. *E. Hewitt.*

Nagata, Jun-iti. A theorem of dimension theory. Proc. Japan Acad. 32 (1956), 166–170.

The main theorem asserts that a T_1 space R is metrizable and of (covering) dimension dim $R \leq n$, if and only if it has a sequence of open coverings $\mathfrak{B}_m$ $(m=1, 2, \cdots)$ for which each $\mathfrak{B}_{m+1}$ is a star-refinement of $\mathfrak{B}_m$, the stars $\mathrm{St}(p, \mathfrak{B}_m)$ form a neighborhood basis for each $p \in R$, and each set of $\mathfrak{B}_{m+1}$ meets at most $n+1$ sets of $\mathfrak{B}_m$. The necessity is fairly immediate; the sufficiency is deduced by an elaborate construction from another characterization of dimension: if R is metric, dim $R \leq n$ if and only if there exist $n+1$ sequences $\mathfrak{U}_m{}^i$ $(1 \leq i \leq n+1,$ $m=1, 2, \cdots)$ of collections of disjoint open sets, with $\mathfrak{U}_{m+1}{}^i$ refining $\mathfrak{U}_m{}^i$, which together form a basis of open sets. This follows easily from the known theorem [M. Katětov, Czechoslovak Math. J. 2(77) (1952), 333–368 MR 15, 815; K. Morita, Math. Ann. 128 (1954), 350–362; MR 16, 501] that, for metric R, dim $R \geq n$ if and only if R is a union of $n+1$ zero-dimensional spaces. {It is not clear to the reviewer that $\bigcup_i \mathfrak{S}_m{}^i$ covers R (p. 170, line 7*), but the difficulty disappears if the definition of $S(U)$, p. 169, line 7*, is modified to include U itself. On p. 170, the proof of (ii) needs slight adjustment in the extreme cases $j=p-1$, $p=0$, $q=0$.} *A. H. Stone* (Manchester).

Nagata, Jun-iti. On uniform topology of functional spaces. II. J. Inst. Polytech. Osaka City Univ. Ser. A. 6 (1955), 71–77.

This paper improves the results of an earlier communication [same J. 5 (1954), 87–95; MR 16, 607]. Let R be a uniform space with uniformity $\{\mathfrak{U}_\alpha\}_{\alpha \in A}$ (=system of entourages). For each $\alpha \in A$ and $x \in R$, let $U_\alpha(x)=\{y: y \in R, (y, x) \in \mathfrak{U}_\alpha\}$. Let $F(R)$ be a family of functions defined on various subsets of R and with values in a uniform space R' having uniformity $\{\mathfrak{B}_{\alpha'}\}_{\alpha' \in A'}$. Let D_f be the domain of $f \in F(R)$. For $f \in F(R)$, $\alpha \in A$, and $\alpha' \in A'$, let $U_{\alpha\alpha'}(f)=\{g\colon g \in F(R),\ \forall x \in D_f\ \exists y \in U_\alpha(x)$ such that $g(y) \in V_{\alpha'}(f(x))$; and $\forall x \in D_g\ \exists y \in U_\alpha(x)$ such that $f(y) \in V_{\alpha'}(g(y))\}$. Theorem 1. The sets

$$\mathfrak{U}_{\alpha\alpha'}=\{(f, g): f, g \in F(R), g \in U_{\alpha\alpha'}(f)\}$$

are a uniformity for $F(R)$.

Let $C(R)$ be the set of all bounded uniformly continuous real functions on R, with the usual order and the above uniform structure. Let $C'(R)$ be the subset of $C(R)$ consisting of all functions with values in $[0, 1]$. The main result of the present paper is as follows. Let R_1 and R_2 be complete uniform spaces. Then R_1 and R_2 possess isomorphic uniform structures if and only if there is a one-to-one mapping of $C(R_1)$ onto $C(R_2)$ that is a uniform structure isomorphism of $C(R_1)$ onto $C(R_2)$ that either preserves the usual order of functions or is a ring-isomorphism of $C(R_1)$ onto $C(R_2)$. A similar result holds for the spaces $C'(R_1)$ and $C'(R_2)$. *E. Hewitt.*

Ishii, Tadashi. On semi-reducible measures. II. Proc. Japan Acad. 32 (1956), 241–244.

The author extends results of a previous note [same Proc. 31 (1955), 648–652; MR 17, 720]. Terminology is as in the review cited. A cardinal number $\mathfrak{m}$ is said to have measure zero (2-valued measure zero) if there is no non-trivial countably additive real-valued measure (2-valued measure) on all subsets of a set of cardinal number $\mathfrak{m}$. Theorem 1. Let X be a completely regular space admitting some complete uniform structure compatible with the topology of X. The following conditions on X are equivalent. (1) Every closed discrete subset of X has a cardinal number with measure zero. (2) Let μ be a Baire measure on X, and let $\{G_\alpha\}$ be a discrete family of open subsets of X such that $\mu(G_\alpha)=0$ for all α. Then $\mu(\bigcup G_\alpha)=0$. (3) Let μ be a Baire measure on X such that for all $p \in X$, there is an open Baire neighborhood $U(p)$ for which $\mu(U(p))=0$. Then $\mu=0$. (4) Every Baire measure on X is semi-reducible in the sense of Katětov. A similar theorem holds for 2-valued measures. As a corollary, the author shows that a space with some complete uniform structure is a Q-space if and only if every discrete closed subspace is a Q-space. *E. Hewitt* (Seattle, Wash.).

Hanai, Sitiro. On closed mappings. II. Proc. Japan Acad. 32 (1956), 388–391.

[For part I see same Proc. 30 (1954), 285–288; MR 16, 275.] Suppose throughout that f is a closed continuous mapping of a normal T_1 space S onto a (necessarily normal T_1) space E. The author proves the following theorems. If S is countably paracompact (he imposes a further restriction on f, but does not seem to use it), then so is E. If for each $p \in E$ the set $f^{-1}(p)$ is compact (or, more generally, has a compact frontier in S), and S is locally compact and has the star-finite property, the same is true of E. {Here a simple proof seems to have been overlooked; E will be locally compact and paracompact, and the star-finite property follows at once.} Conversely, suppose $f^{-1}(p)$ is compact for each $p \in E$; then if E is countably paracompact, or paracompact, or pointwise paracompact, or if E has the star-finite property, then the same is true of S. *A. H. Stone* (Manchester).

Dowker, Yael Naim. On minimal sets in dynamical systems. Quart. J. Math. Oxford Ser. (2) 7 (1956), 5–16.

Let T be a homeomorphism of a compact metric space X onto itself. X is called T-connected if there is no proper closed subset A whose interior contains TA. A minimal set M is called tailed if there is a point $p \notin M$ such that M is the only minimal set contained in the ω-limit set of p. The closed invariant subsets of X are a closed subspace S of the compact space of closed subsets of X, relative to the Hausdorff metric H. The minimal sets constitute a topologically complete subspace Ω of S. A second metric is introduced in S which makes S and Ω complete and which is topologically equivalent to H in Ω, but not in S. These results together with some from an earlier paper with Friedlander [Proc. London Math. Soc. (3) 4 (1954), 168–176; MR 15, 889] are used to show that if no minimal set is both open and closed, or if there is a point that belongs to no minimal set, then either X contains uncountably many minimal sets or else it contains a tailed minimal set. If X is T-connected then any isolated minimal set $M\neq X$ is tailed, hence the number of tailed minimal sets is equal to the number α of minimal sets in case $1<\alpha\leq\aleph_0$. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Gorman, Charles David. A note on recurrent flows. Proc. Amer. Math. Soc. 7 (1956), 142–143.

Let f: $R\times\{t|t \text{ real}\}\to R$ be a continuous flow in a metric space R. f is said to be recurrent if for every two real positive numbers ε and s there exists a number $t>s$ such that $\rho(x, f(x, t))<\varepsilon$ for all $x \in R$.

The author proves the following theorem. If R is compact the following three conditions are equivalent: (i) f is recurrent. (ii) There exists an increasing sequence of numbers $\{t_n\}$, $t_n \underset{n}{\nearrow} \infty$ such that $\lim_{n\to\infty} f(A, t_n)=A$ for every closed set A of R. (iii) There exists an increasing sequence of numbers $\{t_n\}$, $t_n \underset{n}{\nearrow} \infty$ such that

$$\limsup_{n\to\infty} f(A, t_n)\subset A$$

for every closed subset A of R. In general (R not necessarily compact) (I)$\Rightarrow$(II) and (II)$\Rightarrow$(III). *Y. N. Dowker* (London).

Volpato, Mario. Sugli elementi uniti delle trasformazioni funzionali continue. Rend. Sem. Mat. Univ. Padova 25 (1956), 343–356.

Let X be the space of real valued continuous functions defined over an n dimensional bounded closed interval and let T be a continuous function taking closed convex subset $X_0\subset X$ into itself. The basic theorem states that if T satisfies a generalized type of Lipschitz condition, too involved to state here in detail, there exists a non-void closed convex subset $X_0^*\subset X_0$ such that $V_0^*=T(X_0^*)\subset X_0^*$, V_0^* is bounded and T is completely continuous on X_0^*. This shows in particular that the function $T:X_0\to X_0$ has a fixed point. A similar theorem is stated for the space of real functions which are integrable on an n dimensional closed bounded interval. These theorems are applied to proving the existence and uniqueness of a solution $z(x, \lambda)$ of the non-linear integral equation $z(x, \lambda)=\lambda+\int_a^x f[t, z(t, \lambda)]dt$, where the solution $z(x, \lambda)$ and the kernel function $f(x, y)$ are assumed to satisfy certain growth conditions with respect to each variable. *R. E. Fullerton* (College Park, Md.).

See also: Ribenboim, p. 187; Weston, p. 221; Reissig, p. 212; Douglas, p. 236; Choquet, p. 219; Block and Rosenbloom, p. 235.

Algebraic Topology

Wu, Wen-tsün. On Pontrjagin classes. III. Acta Math. Sinica 4 (1954), 323–346. (Chinese. English summary)

We determine completely the Pontrjagin squares in the Grassmannian manifold. As a consequence, we prove that the Pontrjagin classes, reduced mod 4, and hence also mod 12, when combined with the preceding one in this series of papers, of a closed differentiable manifold are topological invariants of that manifold. *Author's summary.*

Borsuk, K.; and Kosiński, A. On connections between the homology properties of a set and of its frontier. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 331–333.

Let $A\subset B\subset Y$, where A, B are compact, and use Čech theory with compact coefficients. Let j: $A\to B$ and i: Bdry $A\to A$ be the injections. The authors prove the following three theorems: (1) If A is deformable, in Y, into $\overline{Y-A}$, then $H_k(\text{Bdry } A)=H_k(A)\oplus\text{Kern } i^+$ for every k.

(2) If $H_k(B)=H_{k+1}(B)=0$, then $H_k(\text{Bdry } A)=H_k(A)\oplus \text{Kern } i^+$. (3) If $j^+=0$, then i^+ is onto. *J. Dugundji.*

Mizuno, Katuhiko. On factor set of the third obstruction. Proc. Japan Acad. 31 (1955), 414–417.

In this note the author considers the third obstruction to mapping a space X into a space Y with three non-vanishing homotopy groups. He is gradually approaching the consideration of obstruction theory from the point of view of Postnikov system [Dokl. Akad. Nauk SSSR (N.S.) **76** (1951), 359–362, 789–791; **79** (1951), 573–576; MR **13**, 374, 375] with which he does not seem familiar. In doing this he generalizes results of Eilenberg and MacLane [Ann. of Math. (2) **60** (1954), 513–577; MR **16**, 392]. The basic point of view to be adopted in such a situation is that Y, or at least its singular complex may be resolved in the following fashion: there is another space Y' a fibre map $f\colon Y\to Y'$ with fibre $K(\pi_r, r)$, and a fibre map $g\colon Y'\to K(\pi_n, n)$ with fibre $K(\pi_q, q)$,w here $n<q<r$. One considers mappings of X into $K(\pi_n, n)$, then the problem of which of these can be lifted to map of X into Y' and then finally which of these can be lifted into Y. If X is a complex, the situation is as follows: 1) A homotopy class of mappings of X into $K(\pi_n, n)$ is uniquely determined by an element of $H^n(X;\pi_n)$. 2) A representative of the element $\alpha\in H^n(X;\pi_n)$ can be lifted to a map of X into Y' if and only if $T_1(\alpha)=0$, where T_1 is the cohomology operation which is the k-invariant of Y', i.e. T_1 corresponds to k^{q+1} the element of $H^{q+1}(\pi_n, n;\pi_q)$ which is the obstruction to a cross section for the fibre map $g\colon Y'\to K(\pi_n, n)$. 3) A map $h\colon X\to Y'$ can be lifted into Y if and only if $h^*(k^{r+1})=0$, where $k^{r+1}\in H^{r+1}(Y', \pi_r)$ is the obstruction to a cross section for the fibre map $f\colon Y\to Y'$.

J. C. Moore (Princeton, N.J.).

Yamanoshita, Tsuneyo. On the unstable homotopy groups of spheres. Proc. Japan Acad. 31 (1955), 610–611.

This paper is a note announcing some results on the homotopy groups of spheres. The author says that he obtains them by the standard technique of killing homotopy groups.

For any space X and prime p, let $\pi_q(X)_p$ denote the quotient of $\pi_q(X)$ by the torsion subgroup consisting of those elements whose order is prime to p. The author proves that

$$\pi_3(S^3)=Z,\ \pi_4(S^3)_2=Z_2,\ \pi_5(S^3)_2=Z_2,\ \pi_6(S^3)_2=Z_4,\ \pi_7(S^3)_2=Z_2,\ \pi_8(S^3)_2=Z_2,\ \pi_9(S^3)_2=0,\ \pi_{10}(S^3)_2=0,\ \pi_{11}(S^3)_2=Z_2,\ \pi_{12}(S^3)_2=Z_2+Z_2,\ \pi_{13}(S^3)_2=Z_4+Z_2,\ \pi_{14}(S^3)_2=Z_4+Z_2+Z_2,\ \pi_{15}(S^3)_2=Z_2+Z_2,\ \pi_{16}(S^3)_2=Z_2,\ \pi_{17}(S^3)_2=Z_2,\ \pi_{18}(S^3)_2=Z_2,\ \pi_{19}(S^3)_2=Z_2+Z_2,\ \pi_{20}(S^3)_2=Z_4+Z_2+Z_2,\ \pi_{21}(S^3)_2=Z_4+Z_2+Z_2.$$

Further he proves that $\pi_q(S^3)_3=0$ for $4\leq q\leq 21$ except for the following values of q, 6, 9, 10, 13, 14, 16, 17, 18, 19, 20, 21, and that for each of these values $\pi_q(S^3)_3=Z_3$.

This work is an extension of work of various authors, particularly H. Toda [C. R. Acad. Sci. Paris **240** (1955), 42–44, 147–149; MR **16**, 846, 847], who had at the time of this paper announced the results of calculating the groups $\pi_{n+r}(S^n)$ for $r\leq 13$. From this brief note it is not possible to see whether or not the author has any new methods of computation. Probably the details of the proof of his results will be quite long. *J. C. Moore.*

Hu, Sze-Tsen. Axiomatic approach to the homotopy groups. Bull. Amer. Math. Soc. 62 (1956), 490–504.

The author gives an exposition of the basic properties of absolute and relative homotopy groups based on the axiomatic approach of Serre and Milnor [Milnor, Ann. of Math. (2) **63** (1956), 272–284; MR **17**, 994].

W. S. Massey (Providence, R.I.).

Jaworowski, J. W.; and Moszyński, K. A theorem on mappings of the sphere into the projective space. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 75–77.

Let S^n be the n-dimensional unit sphere $|x|=1$ and ϕ a continuous mapping of S^n into itself such that $\phi(-x)\neq -\phi(x)$ for every x. Then (1) there is a point x_0 of S^n such that $\phi(-x_0)=\phi(x_0)$, and (2) the degree $d(\phi)$ of ϕ satisfies the condition: $d(\phi)\equiv 0 \pmod 2$ if n is odd, $d(\phi)=0$ if n is even. From (1) it follows that to any continuous mapping f of S^n, $n>1$, into the n-dimensional projective space there is a point x_0 of S^n such that $f(x_0)=f(-x_0)$.

R. H. Fox (Princeton, N.J.).

Clarke, B. A note on Alexander's duality. Mathematika 3 (1956), 40–46.

In this paper the author considers the relationship between the Alexander duality theorem, and the duality of Spanier and Whitehead [Mathematika **2** (1955), 56–80; MR **17**, 653]. Letting S^n denote the n-sphere, a polyhedron K^* is the n-dual of a polyhedron $K\subset S^n$ if K^* is an S deformation retract of S^n-K. If L is a subpolyhedron of K, then it may be assumed that K^* (the dual of K) is a subpolyhedron of L^*.

The author now proves several theorems such as for example the following theorem. Let G, G^* be topological groups orthogonally paired to R_1 (the reals mod 1) one of which is compact, then if $L\subset K\subset S^n$ are finite polyhedra and $K^*\subset L^*\subset S^n$ dual polyhedra there is an orthogonal pairing

$$H_q(K, L; G)\otimes H_{n-q}(L^*, K^*, G^*)\to R_1.$$

J. C. Moore (Princeton, N.J.).

Adams, J. F. On products in minimal complexes. Trans. Amer. Math. Soc. 82 (1956), 180–189.

The fact that the explicit complexes $K(\pi, n)$ of Eilenberg and MacLane [Ann. of Math. (2) **58** (1953), 55–106; **60** (1954), 49–139; MR **15**, 54; **16**, 391] are minimal abelian group complexes has been well-known for many years. The question of determining which minimal complexes may be given the structure of a group complex is unsettled, and is of considerable interest since even at present there seems to be some hope of computing what the structure of a minimal group complex is, e.g. its Postnikov system [Dokl. Akad. Nauk SSSR (N.S.) **76** (1951), 359–362, 789–791; **79** (1951), 573–576; MR **13**, 374, 375].

It is known that any group complex which is abelian has trivial Postnikov invariants, and that if it is minimal it is actually a product of $K(\pi, n)$'s. In this paper the author proves the following theorem which sheds more light on the situation. Theorem: Let X be an associative H-space with unit such that $\pi_r(X)=0$ for $r\neq n, n+1$, and either $n>1$, or $n=1$ and the Postnikov invariant $k^3=0$. Then if SX is the singular complex of X, and M is a minimal subcomplex of SX, M may be given the structure of a group complex so that the diagram

$$\begin{array}{ccc} M\times M & \to & SX\times SX \\ \downarrow & & \downarrow \\ M & \longrightarrow & SX \end{array}$$

is commutative up to homotopy.

This theorem is proved by defining a sort of obstruction to finding an associative product in M, proving that this vanishes, and that under the conditions of the theorem this enables one to prove the existence of an appropriate product in M.

The author also gives an explicit construction of a non-abelian minimal group complex Γ such that $\pi_1(\Gamma)=Z$, $\pi_2(\Gamma)=Z_2$, and $\pi_q(\Gamma)=0$ for $q\neq 1, 2$. This group complex has the property that if $\alpha^2 \epsilon H_2(\Gamma)$ is a generator, then $\alpha^2 \epsilon H_2(\Gamma)$ is a generator.

{In conclusion it may be well to point out that not every associative H-space with unit has the property that a minimal subcomplex of its singular complex may be made into a group complex. In fact there is an example due to J. W. Milnor (unpublished) of an associative H-space X with $\pi_n(X)=Z$, $\pi_{n+1}(X)=Z_2$, $\pi_{n+2}(X)=Z_2$, and $\pi_q(X)=0$ for $q\neq n$, $n+1$, $n+2$ such that there is no minimal group complex with the homotopy type of SX.}

J. C. Moore (Princeton, N.J.).

See also: Griffiths, p. 192; Ozeki, p. 232; Nikaidô, p. 266.

GEOMETRY

Geometries, Euclidean and other

★**Hilbert, David. Grundlagen der Geometrie. Achte Auflage, mit Revisionen und Ergänzungen von Dr. Paul Bernays.** B. G. Teubner Verlagsgesellschaft, Stuttgart, 1956. vii+251 pp. DM 11.40.

Der vorliegende Neudruck ist nicht eine eigentliche Neubearbeitung. Von den Anhängen I–X der siebenten Auflage (B. G. Teubner, Leipzig-Berlin, 1930) wurden nur diejenigen geometrischen Charakters I–V wieder aufgenommen. In Supplement I sind einige früher im Anhang VI angeführte Abhängigkeiten innerhalb des Axiomensystems der reellen Zahlen zusammengestellt. Hinzugekommen ist als Supplement II eine neue vereinfachte Fassung der Proportionenlehre. Supplement III ist im wesentlichen eine Wiedergabe eines Zusatzes des Anhangs II in der fünften Auflage, betreffend die Ableitbarkeit des weiteren Kongruenzaxioms aus dem engeren bei Hinzunahme eines Axioms der Einlagerung. *Aus dem Vorwort.*

Smirnova, H. A. Problem of g-circles. Mat. Sb. N.S. 39(81) (1956), 397–399. (Russian)

Gupta [Proc. Nat. Inst. Sci. India 19 (1953), 315–316; MR 15, 149] introduced the notion of so-called g-circles defined as follows. Given a finite set of non-concyclic points on the plane or sphere any circumference or straight line passing through three and only three points of the set is called a g-circle. Gupta's results were based upon the hypothesis that the number of g-circles is always greater than or equal to 4. The author's principal result is to prove this hypothesis. The method of proof is based upon the consideration of closed convex polyhedra in the sense of A. D. Alexandrov [Convex polyhedra, Gostehizdat, Moscow-Leningrad, 1950; MR 12, 732].

D. K. Kazarinoff (Ann Arbor, Mich.).

Matschinski, Matthias. Décahexaèdre régulier et une propriété spéciale des espaces au nombre de dimensions égal à quatre ou supérieur. C. R. Acad. Sci. Paris 243 (1956), 472–475.

The author has rediscovered J. F. Petrie's regular skew polyhedron {4, 4|4} [Coxeter, Proc. London Math. Soc. (2) 43 (1937), 33–62, pp. 35–43], whose faces are 16 of the 24 squares in the four-dimensional hypercube.

H. S. M. Coxeter (Toronto, Ont.).

Edge, W. L. Conics and orthogonal projectivities in a finite plane. Canad. J. Math. 8 (1956), 362–382.

Any non-singular conic in a finite Desarguesian plane can be put in the form $x^2+y^2+z^2=0$ by appropriate choice of the triangle of reference, if there are $q+1$ points on a line, q being a power of a prime $p>2$. Minus one is a square in GF(q) if $q\equiv 1 \pmod 4$ and a non-square if $q\equiv 3 \pmod 4$. Starting from this the author studies in some detail the relation of the orthogonal group $\Omega(3, q)$ to the geometry of the plane and in particular its relation to the canonical conic above. Of particular interest are the cases $q=5$, 7, 11, since for these values and no others $\Omega(3, q)$ has a permutation representation of degree q. For $q=5$, 7 the q objects permuted may be considered canonical triangles, these being the whole set for $q=5$, and for $q=7$, the members of either one of two imprimitive systems. For $q=11$ the objects may be taken as the Clebsch hexagons. *Marshall Hall, Jr.*

Kremer, Hugo F. On vector components. Soc. Parana. Mat. Anuário 2 (1955), 3–4. (Portuguese)

Barsotti, Leo. Barycentric coordinates in forms of the first order. Soc. Parana. Mat. Anuário 2 (1955), 7–11, 25. (Portuguese)

The purpose of the present article is to set forth the system of barycentric coordinates on the range of points (the straight line as a set of points) with its extensions to pencils of straight lines and of planes, presenting formulas some of which are believed by the author to be hitherto unpublished. *Author's summary.*

Massera, J. L. On the fundamental notions of projective geometry. Bol. Fac. Ingen. Agrimens. Montevideo 5 (1956), 405–458 = Publ. Didact. Inst. Mat. Estadist 1 (1956), 1–56. (Spanish)

Les disciplines mathématiques modernes se présentent comme un ensemble de propositions logiquement déduites de certaines propositions initiales admises (postulats). Cette présentation assure la rigueur, mais l'auteur ne pense pas que l'on doive regarder la science mathématique comme un simple jeu logique à partir d'un système arbitraire de postulats et de définitions. Les postulats et les principales définitions d'une véritable théorie mathématique proviennent de l'examen de la réalité materielle du sujet ou sont naturellement suggérés par le développement interne même de la théorie. L'auteur expose ses idées sur le sujet, en prenant comme exemple la géométrie projective. Il montre comment, partant de la notion intuitive d'espace, et précisant les significations de certaines locutions courantes concernant la géométrie euclidienne, on arrive au système des dix postulats à partir desquels on peut déduire tous les théorèmes de la géométrie projective. Un appendice contient un certain nombre de considérations historiques, et nous fait connaitre les opinions de quelques-uns des géométres qui ont le plus contribué à ériger la géométrie projective en corps de doctrine, et plus particulièrement de Desargues, Descartes, Poncelet et Chasles. *P. Vincensini.*

Lenz, Hanfried. Zur Definition der Flächen zweiter Ordnung. Math. Ann. 131 (1956), 385–389.

Assume that V is a vector space of rank at least three over the not necessarily commutative field F of scalars; and consider a one-to-one mapping σ of the points [= subspaces of rank 1] of V into the space of hyperplanes such that the point B is on the hyperplane $A\sigma$ whenever the point A is on the hyperplane $B\sigma$. The author proves the following interesting theorems: If there exists a line L [= subspace of rank 2] which carries at least two and at most a finite number of points P such that P is in $P\sigma$, then F is commutative; if in addition L carries an infinity of points, then the characteristic of F is $\neq 2$ and σ may be represented by an ordinary bilinear form. The principal step in the proof of these theorems is the proof of the following lemma: If the field F is not commutative, and if σ is an involutorial anti-automorphism of F, then the equation $x+x^\sigma=0$ has an infinity of solutions x in F.

R. Baer (Frankfurt am Main).

Vaccaro, Giuseppe. Proprietà delle superficie degli spazi a 4 e 5 dimensioni in relazione a quadriche. Rend. Mat. e Appl. (5) 13 (1955), 249–269.

The prolongation of kth order of a surface-cap σ_2, of 2nd order in S_r with respect to a hyper-quadric in a S_{r-1}, not passing through the center 0 of σ_2, is the totality of the ∞^2 differential elements E_k which have the center in 0 and belong to the conics through E_2 of the σ_2 and meeting in two points on the aforesaid hyper-quadric.

By the means of this notion of prolongation the author recently proved [same Rend. (5) 14 (1955), 525–532; MR 17, 77] that through a generic point 0 of a surface F in S_3 ten planes pass cutting F in 0 along hyper-osculating conic (with 5-ple intersections) which meet in two points the same conic.

In this paper the author, using the same notion, proves remarkable properties of regular surface-caps in S_4 and in S_5.

C. Longo (Parma).

Al-Dhahir, M. W. Transformational characterizations of commutativity in projective space. Bull. Coll. Arts Sci., Baghdad 1 (1956), 77–81.

Let S_3 be a 3-dimensional projective space with coordinates from an associative division ring R. The author shows that the following are equivalent: (1) R is a field; (2) every incomplete Möbius configuration can be completed; (3) given three non-collinear points P_1, P_2, P_3, and three non-dependent planes p_1, p_2, p_3, such that P_i is on p_i for each i, and the point $p_1p_2p_3$ is on the plane $P_1P_2P_3$ but not on any line P_iP_j, then there exists a unique polarity of S_3, having order two (i.e., a null polarity), mapping each P_i onto p_i; (4) given a tetrahedron and three points on three of its faces, but none on an edge, there exists a unique polarity of S_3 with respect to which the tetrahedron is self-polar and the three points are pairwise conjugate.

D. R. Hughes.

Kummer, H. Translative Zerlegungsgleichheit k-dimensionaler Parallelotope. Arch. Math. 7 (1956), 219–220.

Two sets U, V in k-dimensional euclidean space are called "translationsgleich" ($U\cong V$) if U may be carried on V by a translation. U, V are called "translativ zerlegungsgleich" ($U\approx V$) if there exist decompositions $U=\bigcup_1^n U_\nu$, $V=\bigcup_1^n V_\nu$ such that $U_\nu\cap U_\mu=V_\nu\cap V_\mu=0$ ($\nu\neq\mu$), and $U_\nu\cong V_\nu$ ($1\leq\nu\leq n$). The author proves the following theorem: If U, V are k-dimensional closed parallelotopes, then $U\approx V$ if and only if U, V have the same content. The result is derived from a corresponding theorem by Hadwiger [Collect. Math. 3 (1950), no. 2, 11–23; MR 13, 768] in which $U\approx V$ is defined by means of "elementary" decompositions of U, V into polytopes without common interior points. The main tool of proof is the following lemma: Every closed simplex is "translativ zerlegungsgleich" with its open kernel.

F. A. Behrend (Melbourne).

See also: Jamitzer, p. 268; van Albada, p. 184; Hoffman; Newman; Straus; and Taussky, p. 185; Shephard, p. 191; Linnik, p. 193; Claringbold, p. 244; Archbold and Johnson, p. 244; Istomin, p. 245; Kalynyak, p. 247; Pólya, p. 250; Bhattacharya, p. 251; Isard, p. 265.

Convex Domains, Integral Geometry

Ohmann, D. Ungleichungen für die Minkowskische Summe und Differenz konvexer Körper. Comment. Math. Helv. 30 (1956), 297–304.

Let A, B be convex bodies in E^n and $A\supset B$. Generalizing his result for the plane the author proves with $|x|$ denoting the n-dimensional measure of X that

$$|A+B|+|A-B|\geq(|A|^{1/n}+|B|^{1/n})^n+(|A|^{1/n}-|B|^{1/n})^n,$$

$$|A+B|-|B|\leq(2|A|^{1/n}-|A-B|^{1/n})^n-(|A|^{1/n}-|A-B|^{1/n})^n.$$

These inequalities are stronger than the Brunn-Minkowski inequalities

$$|A+B|^{1/n}\geq|A|^{1/n}+|B|^{1/n},\quad |A-B|^{1/n}\leq|A|^{1/n}-|B|^{1/n},$$

but do not, in contrast to the plane case, constitute a complete system of inequalities for $|A|$, $|B|$, $|A-B|$, $|A+B|$.

H. Busemann (Los Angeles, Calif.).

Hadwiger, Hugo. Über eine vollständige Schar extremaler konvexer Rotationskörper. Jber. Deutsch. Math. Verein. 59 (1956), Abt. 1, 7–12.

Let A denote a convex body of revolution in euclidean k-space. Thus the intersection of A with the $(k-1)$-spaces normal to its axis of revolution are $(k-1)$-spheres. Let a denote their maximum radius. Let W_λ be the λth mixed volume of A with the unit k-sphere and put $w_\lambda=(W_\lambda/\omega)^{1/(k-\lambda)}$, where ω is the volume of the unit k-sphere ($\lambda=0, 1, \cdots, k-1$). The indices ν and μ are fixed; $0\leq\nu<\mu<k$. Through $\alpha=a/w_\mu$, $\beta=w_\nu/w_\mu$, the class of all A's with given a is mapped onto a set $\mathfrak{P}$ of points (α, β) in the plane. The author reviews and announces results which describe $\mathfrak{P}$ completely. They are equivalent to necessary and sufficient conditions for the existence of an A with given a, W_ν and W_μ. The set $\mathfrak{P}$ is bounded, closed, and connected. It is convex in the β-direction. The boundary of $\mathfrak{P}$ consists of the image points of certain elementary A's. For further details the reader is referred to the original paper.

P. Scherk (Philadelphia, Pa.).

Stein, S. The symmetry function in a convex body. Pacific J. Math. 6 (1956), 145–148.

Soit K_n un corps convexe à n dimensions dans E_n, P un point de K_n, $S(P)$ l'intersection de K_n avec son symétrique par rapport à P, V_n la mesure de K_n, $m(P)$ celle de $S(P)$, $f(P)=m(P)V_n^{-1}$; on démontre que: $\int_{K_n} f(P)dP=2^{-n}V_n$, que l'ensemble où $f(P)\geq a$ est convexe et que $f(P)$ atteint son maximum en un seul point; pour

un simplexe K_n, ce maximum est atteint au barycentre et vaut $2(n+1)^{-1}$. *J. Favard* (Paris).

Fejes Tóth, L. Triangles inscrits et circonscrits à une courbe convexe sphérique. Acta Math. Acad. Sci. Hungar. 7 (1956), 163–167. (Russian summary)

The author proves the following theorem, which appeared as a conjecture on page 55 of his book, Lagerungen in der Ebene, auf der Kugel und im Raum [Springer, Berlin-Göttingen-Heidelberg, 1953; MR 15, 248]: Every closed convex curve on the unit sphere has a circumscribed triangle Δ and an inscribed triangle δ whose areas satisfy the inequality

$$\Delta-\delta\leqq 6\{\operatorname{arc}\tan(2\sin\beta)-\beta\}=1.596\cdots,$$

where $\beta=\operatorname{arc}\cos(2^{\frac{1}{2}}-1)/4$. *H. S. M. Coxeter.*

See also: Mahler, p. 196; Volpato, p. 225; Jefimow, p. 229.

Differential Geometry

★ **Jefimow, N. W. [Efimov, N. V.] Differentialgeometrie; Integralgeometrie.** Grosse Sowjet-Enzyklopädie. B. G. Teubner Verlagsgesellschaft, Leipzig, 1956. 44 pp. DM 1.80.

The translation into German is by Viktor Ziegler. The article on differential geometry (35 pages) is from vol. 14, Moscow 1952 (pp. 499–509) and the one on integral geometry from vol. 18, Moscow 1953 (pp. 253–254).

Pogorelov, A. V. On rigidity of convex polyhedra. Har'kov. Gos. Univ. Uč. Zap. 40=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) 23 (1952), 79–89 (1954). (Russian)

This is a new proof for the rigidity of convex polyhedra in the general sense required for A. D. Alexandrov's theory; this means that we allow subdivision of each face by diagonals which do not intersect in the interior of the face and thus admit polyhedra of different combinatorial types for competition. A polyhedral cap is a convex polyhedron with boundary in a plane P and intersected at most once by normals to P. The author shows first that a polyhedral cap is rigid if its boundary is constrained to remain plane. He then proves rigidity of unbounded polyhedra with total curvature 2π, and under an additional condition also when the curvature is less than 2π. Finally he shows rigidity for closed polyhedra. With different proofs these results are also found in A. D. Alexandrov's convex polyhedra [Gostehizdat, Moscow-Leningrad, 1950; MR 12, 732]. *H. Busemann.*

Pogorelov, A. V. Regularity of convex surfaces. Har'kov. Gos. Univ. Uč. Zap. 34=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) 22 (1950), 5–49 (1951). (Russian)

This paper appeared before the author's book "Deformation of convex surfaces" [Gostehizdat, Moscow-Leningrad, 1951; MR 14, 400] which contains the results of the present paper, with the exception of one which is worth mentioning because the problem has interested several mathematicians lately: On a closed convex surface S in E^3 of class C^3 and with positive curvature let $f(R_1+R_2, R_1R_2, u)$ be a function of class C^3 in the variables R_1, R_2, and the unit vector u, where R_1, R_2 are the principal radii of curvature at the point of S with exterior normal u. If $\partial f/\partial R_1\cdot\partial f/\partial R_2>0$, then S is determined by f up to a translation. This was proved in the analytic case by A. D. Alexandrov [C. R. (Dokl.) Acad. Sci. URSS (N.S.) 22 (1939), 99–102]. *H. Busemann.*

Mineo, Massimo. Sulla variazione della curvatura geodetica d'una curva nella rappresentazione d'una superficie su di un'altra. Matematiche, Catania 11 (1956), 1–7.

A. Marussi worked out the relationship between the geodesic curvature of a curve C on a surface Σ and that of C's image formed when Σ is represented on another surface [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 16 (1954), 478–483; MR 16, 745]. In that work tensor methods were used. The present author feels that the tensor analysis is not of special advantage for characterizing the conformal, geometric, and equivalent representations. Without tensor analysis, he arrives easily at essentially the same results for the conformal and geodesic representations, using Tissot's theorem which says that in every real non-conformal representation of one surface on another there is one and only one orthogonal parameter system on the first surface whose image on the second surface is also an orthogonal system. The result reached in the conformal case is here ascribed to Pizzetti [Trattato di geodesia teoretica, Zanichelli, Bologna, 1905. p. 376]. *A. Schwartz* (New York, N.Y.).

Voss, K. Einige differentialgeometrische Kongruenzsätze für geschlossene Flächen und Hyperflächen. Math. Ann. 131 (1956), 180–218.

A parallel map $p\to p'$ of one hypersurface S in E^n on another, S', is defined by the property that the lines pp' are all parallel. If two closed equally oriented convex hypersurfaces S, S' with positive curvature admit a parallel map such that at corresponding points p, p' one of the elementary symmetric functions H_v, v fixed, $v=1, \cdots, n-1$, of the principal curvatures has the same value then $p\to p'$ is (part of) a translation. This theorem remains correct for $H_1=\sum R_i^{-1}$ and non-convex closed oriented hypersurfaces (not necessarily homeomorphic to S^{n-1}) provided the parallel map is regular in terms of the local parameters on S, S' and the shadow boundaries of S, S' corresponding to the direction of pp' contains no interior points. These theorems imply symmetry theorems for single surfaces, for example: A closed hypersurface S of positive curvature and class C^3 has a hyperplane of symmetry normal to the line L, when H_v has the same value at the intersection of S with lines parallel to L.

There are various other theorems, the following is typical: Let F be a closed analytic surface (of arbitrary genus) in E^3 such that a line parallel to one fixed line L intersects F either not at all, or in two points or touches F in one point; moreover a line element where a line parallel to L is tangent to F is not an asymptotic direction on F. Denote the two principal curvatures of F by k_1 and k_2. If then a relation $k_2=f(k_1)$ holds, where $f(x)$ is defined for $x\geqq c$, $f(c)=c$ and $f'(x)<0$ for $x>c$, then F possesses a plane of symmetry normal to L.

H. Busemann (Los Angeles, Calif.).

Wintner, Aurel. On the Hamilton-Jacobi equation of geodesics. Tensor (N.S.) 6 (1956), 1–5.

Consider a positive definite line element

$$ds^2=E(x, y)dx^2+2F(x, y)dxdy+G(x, y)dy^2,$$

where E, F, G are of class C^n, $n\geqq 1$. The first Beltrami

operator $\nabla u=(Gu_x^2-2Fu_xu_y+Eu_y^2)/(EG-F^2)$ is defined for all $n\geqq 1$, but the standard methods yield solutions of $\nabla u=1$ only if $n>1$. For $n=1$ it is not known whether there is any solution. The principal result of this paper is the existence of (a variety of) solutions provided the Gauss curvature of the ds^2 exists and is continuous, which is a weaker condition than $n\geqq 2$. *H. Busemann.*

Bakel'man, I. Ya. Differential geometry of smooth non-regular surfaces. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 2(68), 67–124. (Russian)

This is a systematic investigation of the surfaces

$$x(u, v)=(x_1(u, v),\ x_2(u, v),\ x_3(u, v)$$

in E^3 where the x_i are of class C', $x_u\times x_v\neq 0$, and the $x_i(u, v)$ possess second generalized derivatives in Sobolev's sense which are square summable. Some of the results have appeared earlier [Dokl. Akad. Nauk SSR **82** (1952), 501–504; Uspehi Mat. Nauk (N.S.) **9** (1954), no. 4(62), 155–162; MR **13**, 984; **16**, 622].

The author first discusses properties of Sobolev derivatives, then he shows that his surfaces are surfaces of bounded curvature in A. D. Alexandrov's sense, whose results are therefore valid for the present surfaces. The coefficients L, M, N and the first derivatives of E, F, G exist in Sobolev's sense. Then the standard expressions for the extrinsic Gauss curvature, and the first variation of length remain correct, and so do Codazzi's equations in an integrated form. Gauss' theorem holds in the sense that the extrinsic and intrinsic integral curvatures are equal. The first and the generalized second fundamental forms determine a surface up to motions. The paper concludes by showing that Herglotz' rigidity proof for closed convex surfaces with positive curvature can be extended to the present class of surfaces. *H. Busemann.*

Backes, F. Sur un couple de congruences W déduites d'un réseau conjugué spécial. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 596–607.

L'auteur étudie le problème de la recherche des réseaux conjugués qui sont à invariants ponctuels égaux, à invariants tangentiels égaux, et dont les deux équations de Moutard sont équivalentes. Il montre que ce problème admet une grande variété de solutions, et indique un moyen pour obtenir un nombre étendu de ces solutions à partir de couples de solutions associées de l'équation de Laplace définissant les fonctions harmoniques. Il établit ensuite que les droites d'intersection d des plans osculateurs aux courbes du réseau en un même point O de celui-ci engendrent une congruence W, qu'il en est de même de la droite d' joignant les deux foyers autres que O des deux congruences focales de réseau envisagé, et que les deux congruences (d), (d') forment un couple doublement stratifiable (qui n'est pas le couple le plus général de son espèce). *P. Vincensini* (Marseille).

Busemann, Herbert. Groups of motions transitive on sets of geodesics. Duke Math. J. **23** (1956), 539–544.

This paper is chiefly concerned with G-spaces R [Busemann, The Geometry of Geodesics, Academic Press, New York, 1955; MR **17**, 779] which have the property that, given any two geodesics, there exists a motion of R carrying one to the other. If R is compact, the author showed that R is either a spherical, elliptic, Hermitian elliptic, quaternion elliptic space or the Cayley elliptic plane. For the non-compact case, such spaces R of dimension two and three are determined. The author also discussed the group of motions which is transitive on a dense set of geodesics. To illustrate the difference between compact and non-compact cases, a higher dimensional analogue Q of the quasi-hyperbolic plane was constructed. In Q, there are two classes of geodesics. The total group of motions of Q is transitive on each of these two classes. *H. C. Wang.*

Švec, Alois. Déformations projectives de certaines surfaces a réseau conjugué. Czechoslovak Math. J. **5(80)** (1955), 559–572. (Russian summary)

Une surface F dans l'espace projectif à cinq dimensions est une surface dont les coordonnées d'un point quelconque satisfont à une équation de Laplace

$$x_{uv}=\alpha x+\beta x_u+\gamma x_v.$$

Les courbes paramétriques forment un réseau conjugué, l'unique réseau conjugué de la surface. Etant donné deux surfaces F, il existe toujours une déformation projective d'ordre 2 dans laquelle elles se correspondent, mais le problème de la déformation projective du troisième ordre des surfaces F n'est pas trivial; il est résolu complètement par l'auteur pour les surfaces F_0, solutions de l'équation $x_{uv}=0$.

Ces surfaces F_0 sont telles que leurs premières transformées de Laplace dans les deux sens se réduisent à des courbes; autrement dit, les tangentes aux courbes v (resp. u) le long d'une courbe u (resp. v) forment un cône. La pseudonormale à une surface F en un point x est l'intersection des espaces osculateurs d'ordre trois $[xx_ux_{uu}x_{uuu}]$, $[xx_vx_{vv}x_{vvv}]$ aux deux courbes du réseau qui passent par x, et les surfaces F_0 dont les pseudonormales le long de toute courbe du réseau conjugué forment un cône sont dites surfaces F_1. L'auteur démontre que les surfaces F_0 qui sont déformables sont des F_1 particulières qu'il appelle F_2 et qui sont caractérisées par une propriété géométrique simple; elles dépendent de six fonctions d'une variable. *M. Decuyper* (Lille).

Godeaux, Lucien. Una famiglia di quadriche associata ad una congruenza W. Boll. Un. Mat. Ital. (3) **11** (1956), 137–140.

La représentation des droites de l'espace sur la quadrique Q de Klein de S_5 conduit l'auteur à associer à une congruence W quelconque une suite de quadriques telle que deux quadriques successives se touchent en quatre points caractéristiques. Il étudie les propriétés géométriques de cette suite, et est amené, en particulier, à montrer comment, au rayon générateur de la congruence, on peut attacher intrinsèquement un tetraèdre dont la considération peut être utilisée avec profit dans certaines recherches sur la théorie des congruences W. *P. Vincensini* (Marseille).

Marcus, F. Sur un quadruple et un couple de congruences stratifiables. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), 155–161. (Romanian. Russian and French summaries)

The following two theorems concerning congruences of straight lines in projective 3-space are proved. 1) Let K, K' be two stratifiable congruences of straight lines. Let the associated congruences L, L' be congruences of Waelsch. Then the fundamental pair K, K' describes either congruences W or congruences of Waelsch. 2) If one congruence of a statifiable quadruple is a congruence of Waelsch, then all the congruences of the quadruple are congruences of Waelsch.

This note is based on the ideas of (and frequent reference is made to) one of Finikoff's papers [Rend. Circ. Mat. Palermo, **53** (1929), 313–364]. *R. Blum.*

Cygankov, I. V. Construction of a normal W-congruence. Molotov. Gos. Univ. Uč. Zap. 8, no. 1 (1953), 25–29. (Russian)

Given a surface S, to find a normal W-congruence of which S is a focal surface. In this case S is applicable to a surface of revolution. The equations are derived and applied to the case of a pseudospherical surface.

D. J. Struik (Cambridge, Mass.).

Gallarati, Dionisio. Alcune osservazioni sopra le varietà i cui spazi tangenti si appoggiano irregolarmente a spazi assegnati. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 193–199.

The present paper deals with the differentiable W_k's, of a projective S_r, whose tangent S_k's intersect one or more subspaces of S_r in spaces of given dimensions (greater than usual). Special attention is paid to the cases when the subspaces are independent, when W_k is algebraic, and when $k=2$, $r=4$.

B. Segre (Rome).

Nádeník, Zbyněk. On projective differential invariants of a plane congruence of curves. Časopis Pěst. Mat. 78 (1953), 229–258. (Czech)

On étudie le système complet des invariants différentiels de la congruence plane V des courbes à l'aide des méthodes du repère mobile de E. Cartan. On considère aussi la correspondance nulle qui existe entre les points A et les tangentes des courbes (A) de la congruence V construites dans les points A. En partant de cette correspondence et en utilisant l'environ du 2 ordre on peut adjoindre à chaque point A le faisceau des coniques qui possèdent dans le point A le contact du 3ième ordre et qui touchent la courbe (A) (avec le contact juste du 1èr ordre dans A). On obtient les nouveaux systèmes des courbes en considérant les cas particuliers des repères de la congruence V et on étudie aussi les cas dans lesquels quelques-uns de ces systèmes des courbes sont confondus.

F. Vyčichlo (Praha).

Švec, Alois. Déformation projective des congruences de droites dans S_n. Czechoslovak Math. J. 5(80) (1955), 546–558. (Russian summary)

M. E. Čech a introduit la notion de déformation ponctuelle de congruences [Czechoslovak Math. J. 5(80) (1955), 234–273; MR 17, 658]. Dans le travail actuel, l'auteur résoud le problème de la déformation projective du second ordre des congruences de droites plongées dans l'espace projectif à $n\geqq 4$ dimensions, dans le cas où la congruence possède deux surfaces focales qui portent chacune un seul réseau conjugué. Pour $n\geqq 5$, la déformation projective est équivalente à la déformation ponctuelle. Pour $n\geqq 4$, la situation est un peu différente: deux congruences en déformation projective sont aussi en déformation ponctuelle, mais la proposition réciproque n'est pas vraie. En utilisant une notion de déformation bitangente, l'auteur donne les conditions pour qu'une correspondance ponctuelle entre deux congruences soit une déformation projective. Il montre enfin que dans S_4 les congruences qui sont en déformation projective du second ordre avec une congruence donnée dépendent de huit fonctions d'une variable.

M. Decuyper (Lille).

Muracchini, Luigi. Le trasformazioni puntuali che possegono rette ipercaratteristiche. Boll. Un. Mat. Ital. (3) 11 (1956), 182–188.

Si studiano le corrispondenze puntuali T fra due S_r proiettivi S, S' per le quali in una coppia di punti corrispondenti $(A, \bar{A})$ si verifica il seguente fatto: esiste una retta a per A, tale che ogni curva di S uscente da A e tangente ivi ad a si muta, per la T e per (almeno) una omografia K_0 tangente a T in A, in due curve aventi in A un contatto analitico del 3º ordine. L'A. dà una condizione necessaria (che fa intervenire un sistema di coni cubici di vertice A) affinchè la circostanza voluta si presenti, nel qual caso l'omografia tangente K_0 è unica; dimostra poi che per $r>2$, la condizione data è anche sufficiente se la si suppone verificata per ogni coppia di punti corrispondenti, mentre non lo è per $r=2$.

V. Dalla Volta.

Villa, Mario. Classificazione delle trasformazioni puntuali di 3ª specie fra piani. Boll. Un. Mat. Ital. (3) 11 (1956), 141–149.

L'A. studia le trasformazioni puntuali fra piani proiettivi, tali che, in un generico punto, le tre direzioni caratteristiche coincidano (dette del 3º tipo); la classificazione, basata sulle corrispondenze linearizzate e sulle corrispondenze quadratiche ad esse osculatrici, non differisce da quella ottenuta da L. Muracchini per altra via [Mem. Accad. Sci. Torino. Cl. Fis. Mat. Nat. (3) 1 (1955), 25–44; MR 18, 145].

V. Dalla Volta (Roma).

Speranza, Francesco. Classificazione delle trasformazioni puntuali di 2ª specie fra piani. Boll. Un. Mat. Ital. (3) 11 (1956), 210–216.

In analogia a quanto fatto da Villa [Vedi la precedente recensione] per le trasformazioni puntuali fra piani proiettivi del 3º tipo, si studiano qui le analoghe trasformazioni del 2º tipo (per le quali cioè in un punto generico due rette caratteristiche coincidono) e se ne ottiene una trasformazione in base agli stessi criteri di Villa. La classificazione appare qui più completa di quella data per altra via da Muracchini [Mem. Accad. Sci. Torino. Cl. Fis. Mat. Nat. (3) 1 (1955), 25–44; MR 18, 145].

V. Dalla Volta (Roma).

See also: Zimmermann, p. 199.

Riemannian Geometry, Connections

Okubo, Tanjiro. On the extended plane transformations in the homogeneous contact tensor fields. Tensor (N.S.) 6 (1956), 32–59.

This is a contribution to contact tensor calculus in the form presented by K. Yano and the reviewer [Ann. Mat. Pura Appl. (4) 37 (1954), 1–36; MR 16, 626]. Necessary and sufficient conditions are given for the existence of a homogeneous contact transformation such that the first contact frame vanishes. The particular contact transformations which preserve this property of the vanishing of the first contact frame are of a type to which the author gives the name "extended plane transformations". This result is a dual of a result given for the second contact frame by Y. Muto [Proc. Phys.-Math. Soc. Japan (3) 20 (1938), 451–457]. The author then considers spaces whose metric tensor depends only on the component of a covariant vector p. In the final paragraph the author proves that the only Riemannian spaces admitting homogeneous contacts transformations making the first contact frame vanish are Euclidean spaces and the transformations must be extended plane transformations.

E. T. Davies (Southampton).

Chaki, Manindra Chandra. Some formulas in a Riemannian space. Ann. Scuola Norm. Sup. Pisa (3) **10** (1956), 85–90.

In this paper a number of theorems and formulas involving two arbitrary affine connections in a Riemannian space V_n have been established by imposing certain conditions on the affine connections. In section I it has been assumed that the covariant derivatives of the metric tensor of the V_n with respect to the affine connections are the same, while in Section 2 the torsions of the affine connections have been taken to be the same. (Author's summary.) *V. Hlavatý* (Bloomington, Ill.).

Kuiper, Nicolaas H. Groups of motions of order $\frac{1}{2}n(n-1)+1$ in Riemannian n-spaces. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 313–318.

The following global form of a theorem of Yano [Trans. Amer. Math. Soc. **74** (1953), 260–279; MR **14**, 688] is given in this paper, and proved in a very simple way.

If a Riemannian space V_n, $n>4$, $\neq 8$, admits a connected group G_r of motions of order $r=\frac{1}{2}n(n+1)+1$, the following are all cases: a) $V_n=L\times W_{n-1}$ or $V_n=S^1\times W_{n-1}$, where L is the euclidean line, S^k the k-sphere, and W_{n-1} an $(n-1)$-dimensional euclidean or noneuclidean space or an $(n-1)$-sphere. G_r is the identity component of the group of those motions in V_n that decompose into a motion of L and one of W_{n-1}. a') V_n admits $S^1\times S^{n-1}$ as a double covering; the covering group maps each pair (a, b) into the pair (a', b') of diametric points of S^1 and S^{n-1} respectively. G_r is the group induced from case a). b) $V_n=H_n$, the hyperbolic space, and G_r consists of all limit rotations with fixed infinite center.

The proof is based on very elegant geometric considerations; on a theorem of Montgomery and Samelson [Ann. of Math. (2) **44** (1943), 454–470; MR **5**, 60] from which the transitivity of G_r follows; and on an algebraic theorem of Kuiper and Yano which supplies the algebraic nature of the curvature tensor and another quantity. *A. Nijenhuis* (Seattle, Wash.).

Teleman, C. Sur certains espaces symétriques. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 977–1002. (Romanian. Russian and French summaries)

Riemannian spaces $V_{4p}{}^*$ were introduced by E. Cartan [Ann. Sci. Ecole Norm. Sup. (3) **44** (1927), 345–467] as an example of simply connected and symmetric spaces having, in addition, other remarkable properties.

In the present paper the author proves that these spaces are equivalent to a certain non-holonomous space $V_{4p+3}{}^{4p}$ defined by three given Pfaff equations on the sphere:

$$(X^1)^2+\cdots+(X^{4p+4})^2=R^2.$$

It is shown that the $V_{4p}{}^*$ can be considered as a quaternionic projective space Q^p and the metric of this space is given in terms of p quaternionic variables.

There follows a study of the properties of the geodesics of these spaces in which, to a large extent, the results of E. Cartan are rederived. In addition it is shown that, in a certain coordinate system, the geodesics are plane curves of the second order. It follows that the $V_{4p}{}^*$ is a Kagan space which is $4p-2$ times projective (without being subprojective).

Some results of the present paper have been announced in Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 731–734 [MR **17**, 528]. *R. Blum.*

Tonooka, Keinosuke. On a geometry of three-dimensional space with an algebraic metric. Tensor (N.S.) **6** (1956), 60–68.

In an n-dimensional space $F_n{}^{(p)}$ endowed with a differential form of order p $(\geqq 3)$ $ds^p=a_{\alpha_1\cdots\alpha_p}dx^{\alpha_1}\cdots dx^{\alpha_p}$, with non-vanishing discriminant $\mathfrak{A}$, the theory of invariants was treated only in the special case $n=2$ [cf. A. E. Liber, Trudy Sem. Vektor. Tenzor. Anal. **9** (1952), 319–350; MR **14**, 388]. The author establishes the connection in the space $F_3{}^{(3)}$ and this paper is the first one which deals with the 3-dimensional space endowed with the differential form of order $p\geqq 3$. More precisely he takes into consideration the symmetric tensor density of weight 10 defined by 10 linearly independent equations $a_{\alpha\beta\gamma}B^{\alpha\beta\delta}=0$, $h_{\alpha\beta\gamma}B^{\alpha\beta\gamma}=\mathfrak{A}$, where $h_{\alpha\beta\gamma}$ is the coefficient of the Hessian of the curve $a_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma=0$ in a 2-dimensional projective space. Then on making use of the symmetric tensor density $A^{\alpha\beta\gamma}=\frac{1}{8}\varepsilon^{\alpha\lambda\xi}\varepsilon^{\gamma\nu\zeta}B^{\beta\nu\eta}a_{\lambda\mu\nu}a_{\xi\eta\zeta}$ of weight 12 and the tensor density $L_{\alpha\beta\gamma}$ (symmetric in α and β) of weight 2 defined by the 18 linearly independent equations $A^{\alpha\beta\gamma}L_{\alpha\beta\delta}=0$, $B^{\alpha\beta\gamma}L_{\alpha\beta\delta}=\mathfrak{A}\delta_\delta{}^\gamma$, he introduces the connection parameters

$$\Gamma^\lambda_{\mu\nu}=\frac{1}{3\mathfrak{A}}P^{\lambda\alpha\beta\gamma}_\mu\partial a_{\alpha\beta\gamma}/\partial x^\nu$$

which satify $P^{\lambda\alpha\beta\gamma}_\mu\nabla_\nu a_{\alpha\beta\gamma}=0$, and proves its uniqueness, where $P^{\lambda\alpha\beta\gamma}_\mu$ is determined by the equations $P^{\lambda\alpha\beta\gamma}_\mu a_{\nu\beta\gamma}=\mathfrak{A}\delta_\mu{}^\alpha\delta_\nu{}^\lambda$, $P^{\lambda\alpha\beta\gamma}_\mu L_{\alpha\beta\gamma}=0$. He names such a connection the metric one. It is proved that in $F_3{}^{(3)}$ there exists no metric connection except when $F_3{}^{(3)}$ is flat, that is, when we can choose such a coordinate system that $\partial a_{\alpha\beta\gamma}/\partial x^\nu=0$. At the last he states the relation between the connection induced in $F_2{}^{(3)}$ immersed in $F_3{}^{(3)}$ by that of $F_3{}^{(3)}$ and the intrinsic connection of the $F_2{}^{(3)}$ itself. *A. Kawaguchi* (Sapporo).

Ozeki, Hideki. Infinitesimal holonomy groups of bundle connections. Nagoya Math. J. **10** (1956), 105–123.

The present paper is an extension of papers of Ambrose and Singer [Trans. Amer. Math. Soc. **75** (1953), 428–443; MR **16**, 172] and Nijenhuis [Nederl. Akad. Wetensch. Proc. Ser. A. **56** (1953), 233–240, 241–249; **57** (1954), 17–25; MR **16**, 171, 172]. The first paper gives expressions for the Lie algebra of the holonomy group of a connection in a differentiable fiber bundle; the latter deals only with the case of bundles of linear spaces in a somewhat different terminology and approach, and in addition to giving the Ambrose-Singer result for this case, defines and develops the concepts of local holonomy group and infinitesimal holonomy group. The paper under review first refines the Ambrose-Singer result, and then follows the line of the Nijenhuis paper, but now deals with the case of a general differentiable fiber bundle with Lie structure group.

Let $P=P(M, G, \pi)$ be a principal bundle in which a connection is given by means of a field of "horizontal" planes satisfying the usual conditions, and let Ω be the $L(G)$-valued curvature form. For $x\in P$, m_x is the vector space $CL(G)$ spanned by the values of Ω on any pair of vectors tangent to P at x. Let $P(U, x)$ represent the set of points $y\in P$ that can be connected to x by a horizontal piecewise differentiable curve whose projection on M lies in the open set UCM (with $\pi(x)\in U$). The holonomy group $H(x)$ is the set of all $a\in G$ such that $x\cdot a\in P(M, x)$; the restricted holonomy group $H^0(x)$ corresponds to curves whose projection on M is homotopic to a point

($H^0(x)$ is the identity component of $H(x)$), and more generally, $H(U,x)=\{a \in G | x\cdot a \in P(U,x)\}$. The refined version of the Ambrose-Singer theorem states that $L(H^0(x))$ is spanned by the union of all m_y for $y \in P(x)$. Following Nijenhuis the local holonomy group $H^*(x)$ is defined as the intersection of all $H(U,x)$, where U runs over all open neighborhoods of $\pi(x)$. It is easily possible to define covariant differentiation processes for the curvature form, and the author proves that the vector space $\mathfrak{h}'(x) \subset L(G)$ spanned by the values of Ω and its covariant derivatives of all orders at x, is a Lie algebra; and that $\mathfrak{h}'(x) \subset L(H^*(x))$. The infinitesimal holonomy group $H'(x)$ is by definition the connected subgroup of G whose Lie algebra is $\mathfrak{h}'(x)$. Obviously, $H'(x) \subset H^*(x) \subset H^0(x) \subset H(x)$. Typical results for the various holonomy groups are: (1) The function $\dim H^*(x)$ is upper semi continuous; (2) if $\dim H^*(x)$ is constant, then $H^0(x)=H^*(x)$; (3) the function $\dim H'(x)$ is lower semi continuous; (4) if $\dim H'(x)=\dim H^*(x)$ at one point, then $H'(x)=H^*(x)$ in a neighborhood of that point; (5) if $\dim H'(y)$ is constant in an open neighborhood of a point x, then $H^*(x)=H'(x)$; (6) if $\dim H'(x)$ is constant, then $H^0(x)=H'(x)$; (7) if all spaces and the connection are analytic, then $\dim H'(x)$ is upper semi-continuous, and therefore constant; hence $H^0(x)=H'(x)$. — The extension of all these results from the linear case of Nijenhuis to the general case require some refinements in the analysis, which are given in a quite elegant way. The author even corrects some three errors in the Nijenhuis paper. In the last section on Cartan connections the author follows the line of Kobayashi [Thesis, Univ. of Washington, 1956] and establishes expressions for the Lie algebra of the infinitesimal holonomy group. The specialization to affine connections (already given in the Nijenhuis paper) is left for a future occasion. *A. Nijenhuis* (Seattle, Wash.).

Cossu, Aldo. Sulle connessioni tensoriali integrabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 258–264.

Cossu, Aldo. Connessioni tensoriali per tensori doppi misti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 421–427 (1956).

These two papers deal with connections for tensors ξ^{ik} and $\xi_k{}^i$ respectively; the covariant differentials being $d\xi^{ik}+L^{ik}_{rst}\xi^{rs}dx^t$ and $d\xi_k{}^i+E^{is}_{rkt}\xi_s{}^r dx^t$ respectively. The first paper is mainly concerned with the case where the associated curvature tensor vanishes; in particular the situation where a basis for the covariant constant tensor fields is formed by products of vectors:

$$\underset{\alpha\beta}{\xi}{}^{ik}=\underset{\alpha}{\zeta}{}^i\underset{\beta}{\eta}{}^k;\ \alpha,\beta=1,\cdots,n;\ \det|\underset{\alpha}{\zeta}{}^i|\neq 0,\ \det|\underset{\beta}{\eta}{}^k|\neq 0.$$

The second paper establishes among others a torsion-type tensor whose vanishing is the necessary and sufficient condition that inner products $\xi_k{}^i\eta_i{}^k$ are invariant under parallel displacements of $\xi_k{}^i$ and $\eta_i{}^k$. The tensors ξ^{ik}, $\xi_k{}^i$ establish respectively correlations and collineations in the projective $(n-1)$-space at infinity of the tangent spaces. Conditions are established that these mappings are invariant under the connections. Various other similar problems are considered. *A. Nijenhuis*.

Kano, Chôtaro. Conformal geometry in an n-dimensional space with the arc length $s=\int\{A_i(x,x')x''^i+B(x,x')\}^{1/p}dt$. Tensor (N. S.) **5** (1956), 187–196.

The main purpose of the paper is to give the foundation to the conformal geometry in a special Kawaguchi space with the arc length stated in the title [cf. Kawaguchi, Proc. Imp. Acad. Tokyo **12** (1936), 205–208; Trans. Amer. Math. Soc. **44** (1938), 153–167]. At the first, under the conformal transformation in the space such that $F=e^{p\rho}F$, where $F=A_ix''^i+B$ and ρ is a function of x^i and x'^i, the rule of variance of the fundamental quantities A_i, B, G_{ij}, A_{ij} and Γ^i, and then the conformally invariant base connection is introduced on making use of the rule. The author gives in § 2 the conformal absolute differentials and derivatives of a contra- or covariant vector and of a relative vector. Since the conformal derivatives of a relative vector are not invariant under a transformation of the parameter t, in § 3 he normalizes it to achieve invariance, i.e. the intrinsic absolute differential of a relative vector is derived. In § 4 the conformal curvature tensors are calculated and the relations among them are found. The necessary and sufficient condition that any space be conformal to a space with vanishing conformal curvature tensors is stated. *A. Kawaguchi* (Sapporo).

Kawaguchi, Syun-ichi; and Nobuhara, Tetsuro. On extremal curves in a special Kawaguchi space. Tensor (N.S.) **5** (1956), 197–200.

In a special Kawaguchi space with the metric $ds=(A_ix''^i+B)^{1/p}dt$, there are two kinds of intrinsic curves, i.e. affine paths satisfying the differential equation $x''^i+2\Gamma^i=x'^i$ and extremal curves along which the Euler vector of the metric vanishes [cf. Kawaguchi, Proc. Imp. Acad. Tokyo **12** (1936), 205–208; Trans. Amer. Math. Soc. **44** (1938), 153–167]. On making use of the same method as that used by the reviewer, the following theorems are proved. 1) A condition that an affine path be extremal is $A_hK_{ki}{}^h x'^k=0$. 2) If $p=1$ and $A_hK_{ki}{}^h x'^k=0$, an extremal curve is always an affine path. Thereupon $K_{ki}{}^h=R_{kij}{}^h x'^j$ and $R_{kij}{}^h$ is the curvature tensor of the space.

A. Kawaguchi (Sapporo).

Algebraic Geometry

Tyrrell, J. A. Complete quadrics and collineations in S_n. Mathematika **3** (1956), 69–79.

This paper contains a concise general formulation of basic properties of complete quadrics and complete collineations of S_k. Earlier papers by Study, Severi and van der Waerden have dealt in detail with the case of complete conics in S_2, and more recently special studies have been made by A. R. Alguneid [see MR **14**, 681; **15**, 343] and the reviewer [see MR **10**, 472; **14**, 78] of complete quadrics in S_3 and S_4 and of complete collineations in S_2, S_3 and (with less detail) S_n. The basic theories for quadrics and collineations are very similar, each starting from a matrix $\mathbf{a}$, of order $n+1$, symmetric or unsymmetric, with elements a_{ij} which are all independent indeterminates. The respective models Ω and W of complete quadrics and of complete collineations of S_n are each defined by a generic point whose coordinates are all the polynomials in the a_{ij} of the form $p_1p_2\cdots p_n$, where p_α ($\alpha=1,\cdots,n$) is any element of the adjugate $\mathbf{a}_{(\alpha)}$ of $\mathbf{a}$ (with $\mathbf{a}_{(1)}=\mathbf{a}$). This approach entails an exhaustive classification of all types of degenerate quadric or collineation that can correspond to all possible specializations of the generic point on Ω or W. It is proved that each of Ω and W is non-singular, that each has n non-singular primary degeneration manifolds, and that these intersect regularly in all the higher degeneration manifolds; also

that the complete quadrics and collineations of S_n are thereby classified in each case into 2^n projectively distinct types. By use of the method of degenerate collineations, in particular by using collineations with matrices of the type diag $[\varepsilon^n, \varepsilon^{n-1}, \cdots, 1]$, the author establishes bases for sub-manifolds of the highest dimension on Ω and W; and attention is drawn in this connection to the special role of "Halphen systems" (of quadrics or collineations) which are those containing every complete quadric or collineation of the most degenerate type. *J. G. Semple.*

Lang, Serge. On the Lefschetz principle. Ann. of Math. (2) **64** (1956), 326–327.

Le principe de Lefschetz affirme que, si une assertion algébro-géométrique de type usuel est démontrée lorsque le domaine universel est le corps des nombres complexes (par exemple par une méthode analytique ou topologique), cette assertion est vraie pour tout domaine universel de caractéristique 0. L'auteur donne ici un exemple d'un autre genre: un théorème, qu'il a démontré par des méthodes arithmétiques dans le cas de la géométrie algébrique sur un corps fini, et qu'il étend ici au cas d'un corps de base quelconque au moyen de la théorie de la réduction modulo p de Shimura [Amer. J. Math. **77** (1955), 134–176; MR **16**, 616]. Il s'agit du théorème suivant: tout revêtement abélien non ramifié d'une courbe complète C sans singularités est induit sur C par un revêtement $\lambda : A \to J$ de la jacobienne J de C, où A est une variété abélienne et λ une isogénie séparable.

P. Samuel (Clermont-Ferrand).

Wallace, Andrew H. Tangency and duality over arbitrary fields. Proc. London Math. Soc. (3) **6** (1956), 321–342.

The starting-point of this lucid and interesting paper is the observation that one cannot guarantee that the point of contact of a generic tangent to an algebraic curve is unique if the ground field is arbitrary. That is to say, over a field of finite characteristic every tangent to a curve may be a multiple tangent. This leads to a reexamination of the classical theory of tangential loci. The main result of this work is that if V is a variety, and V' denotes the variety of tangent hyperplanes, then, in the first place it is not in general true that V is the variety of tangent hyperplanes to V'. In fact the symmetric duality of the classical case does not apply. Again, it is not in general true that a generic tangent hyperplane to V touches V along a linear variety, as in the classical case.

In the course of the paper the implicit function theorem of algebraic geometry is proved, the main point being to show the effect of finite characteristic of the ground field. Again, the equation of an algebraic variety is put into a form which exhibits the differences between the classical and the general case. The paper ends with illustrations of the theory, and suggestions for further investigation.

D. Pedoe (Khartoum).

Fernández, Joaquín Arregui. Arithmetic study of geometry on an algebraic curve. Rev. Acad. Ci. Madrid **50** (1956), 11–17. (Spanish)

È ben noto il "metodo rapido" di F. Severi per lo studio della geometria sopra una curva algebrica; metodo rielaborato recentemente dallo stesso Severi [Pont. Acad. Sci. Acta **14** (1951), 143–152; MR **14**, 80] e da E. Vesentini [ibid. **15** (1953), 137–152; MR **15**, 343]. Nella nota presente, l'Autore, ponendosi dal punto di vista dell' algebra astratta, riprende le idee di F. Severi e giunge ad una dimostrazione intrinseca del teorema di Riemann e Roch. *D. Gallarati* (Genova).

Nollet, Louis. Sur les genres pseudocanoniques des surfaces algébriques régulières dans leurs rapports avec la structure algébrique du premier groupe de torsion. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 579–595.

Soit S une surface algébrique irréductible et régulière, sans points multiples ni courbes exceptionnelles de première espèce. On considère sur S les systèmes linéaires pseudocanoniques $|K+Z|$, où K est une courbe canonique et Z un diviseur algébrique de zéro. On sait que les diviseurs Z forment un module fini T que l'on appelle le premier groupe de torsion de S. L'A. démontre ici que la dimension des différents systèmes $|K+Z|$ n'est pas toujours p_g, comme l'on croyait. On peut le montrer sur un exemple; il suffit à cet égard de supposer que S possède un faisceau $\mathcal{E}$ de courbes elliptiques irreductibles E de degré 0. Si les éléments réductibles de $\mathcal{E}$ sont e_iE_i ($i=1, \cdots, s$ fini), on a:

$$K \equiv (p_g-1)E + \sum (e_i-1)E_i;\quad Z \equiv \sum z_iE_i - zE;$$

et $z = \sum z_i/e_i$ où les entiers z_i sont tels que $0 \leq z_i < e_i$. En partant de ces formules et en supposant que $s' \leq s$ des z_i soient positifs, on trouve que la dimension effective de $|K+Z|$ vaut $p_g-1-z+s'$; et pour que toutes les surabondances $s'-z-1$ soient nulles il faut et il suffit que le groupe T soit isomorphe à un groupe G_4 de Klein ou bien à une groupe cyclique d'ordre p^ν où p est une nombre premier. Si l'on ne se restreint plus aux surfaces donées d'un faisceau de courbes elliptiques, alors les conditions précédentes pour que tous les systèmes $|K+Z|$ soient réguliers sont seulement nécessaires. *E. Togliatti.*

Roth, L. Irregular threefolds which possess anticanonical systems. Proc. Cambridge Philos. Soc. **52** (1956), 617–622.

The following results are established. Suppose that an irregular threefold V contains an irreducible anticanonical system $|A|$, if dimension ≥ 2, and free from base points. Then (1) the generic A is a Picard surface, (2) V contains a rational congruence of elliptic curves, of which $|A|$ is compounded, and an elliptic pencil of rational surfaces unisecant to the congruence, (3) the geometric genus and the plurigenera of V are all zero, while the arithmetic genus and the superficial irregularity are each equal to unity. Further, if the elliptic pencil is free from fundamental surfaces, then V is an elliptic threefold of determinant unity. The author suggests that, if there are fundamental surfaces, it is probable that they are exceptional, i.e. birationally transformable into the neighbourhoods of simple points or curves on some birational transform of V. *J. A. Todd* (Cambridge, England).

Gigl, Helmut. Über die Multiplizität eines isolierten Schnittpunktes von n Hyperflächen im R_n. Monatsh. Math. **60** (1956), 198–204.

The author offers another proof of the theorem to the effect that if the origin is an m_i-fold point of the hypersurface $f_i(x_1, x_2, \cdots, x_n)=0$ ($i=1, 2, \cdots, n$) and if it is an isolated point of intersection of these n varieties, then the intersection multiplicity at the origin is not less than $m_1m_2\cdots m_n$. {This result was proved in 1937 by Zariski [Trans. Amer. Math. Soc. **41** (1937), 249–265], and other proofs have been given more recently. In particular, this theorem is a special case of a theorem given by D. G. Northcott [Quart. J. Math. Oxford Ser. (2) **4** (1953), 67–80; MR **14**, 941] as an illustration of the general theory

which he developed in his paper. The methods used by the present author are the same as those of Northcott.}
H. T. Muhly (Iowa City, Ia.).

See also: Gallarati, p. 231.

NUMERICAL ANALYSIS

★**Hausholder (Householder), A. S. [А. С. Хаусхолдер.] Основы численного анализа. [Principles of numerical analysis.]** Izdat. Inostr. Lit., Moscow, 1956. 320 pp. 14.65 rubles.

This is a translation of the book reviewed in MR 15, 470.

★**Łukaszewicz, Józef; and Warmus, Mieczysław. Metody numeryczne i graficzne. Część I. [Numerical and graphical methods. Part I.]** Państwowe Wydawnictwo Naukowe, Warszawa, 1956. 429 pp. (1 plate). zł. 29.20.

This is a well-written text-book on standard numerical and graphical methods which is suitable for a beginner who had no special training in the subject. The author's aim was to fill an obvious gap in the Polish mathematical literature. It does, nevertheless, contain the authors' original contributions to the art of computing. Since there is an almost complete absence of references, these contributions are difficult to spot and to appraise. The book describes an interesting variant on matrix calculus developed by T. Banachiewicz, and called the "calculus of Cracovians" together with its application to the solution of systems of linear equations.

The emphasis is on procedures, rather than on proofs or discussion. Here is a list of topics: Theory of maximum error; Calculus of differences; Interpolation; Approximation; Algebraic equations; Linear equations and Cracovians; Numerical and graphical differentiation and integration; Co-ordinates and graphs; The slide-rule; Nomograms.

This useful booklet ends with a very modest set of tables and contains an Index. *J. Kestin.*

Numerical Methods

Nečepurenko, M. I. On the convergence of approximate methods. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 704–706. (Russian)

It is remarked that questions of convergence arising in solving the equation $\phi(x)=0$ by the iterative procedure $x_{n+1}=x_n-\Delta(x_n)$ can be reduced to similar questions in relation to the application of Newton's method

$$z_{n+1}=z_n-[\psi'(z_n)]^{-1}\psi(z_n)$$

to an auxiliary equation $\psi(z)=0$. Such a reduction is possible if one can find a function ψ such that $\Delta(x)=[\psi'(x)]^{-1}\psi(x)$. In the case of real or complex scalars it is sufficient to choose

$$\psi(x)=\exp\left[\int_{x_0}^{x}\frac{dx}{\Delta(x)}\right].$$

Conditions of convergence given by L. V. Kantorovič [same Dokl. (N.S.) **59** (1948), 1237–1240; MR **9**, 537] for Newton's method are then expressed in terms of Δ for this case. *R. G. Bartle* (Urbana, Ill.).

Hahn, F. H. Linear programming. Research **9** (1956), 260–267.

Expository paper.

★**Bodewig, E. Matrix calculus.** North-Holland Publishing Company, Amsterdam, 1956. xii+334 pp. 26.50 guilders.

Both title and preface of the book are misleading. They hide the fact that the bulk of the book consists of very useful discussions about systems of linear equations, matrix inversion and eigenvalue problems, all from a numerical point of view, with special emphasis on the requirements made by modern calculating equipment. Many different methods are explained and compared, with elaborate discussions as to their efficiency. In many cases the number of necessary operations is studied. Most discussions are closed by summaries, stating the main conclusions, and intended for swift consultation. Several methods are illustrated by numerical examples. A special mention may be made of a practical section about geodetic matrices.

Much of the material embodied in this book was given in previous reports [Nederl. Akad. Wetensch., Proc. **50** (1947), 49–57, 930–941, 1104–1116, 1285–1295; **51** (1948), 53–64, 211–219; Atti Sem. Mat. Fis. Univ. Modena **4** (1950), 133–193; **5** (1951), 3–39; MR **8**, 407; **9**, 250, 382, 621; **13**, 991; **14**, 1128].

In his preface, the author says: "The aim of this book is a systematic calculation with the true building blocks of a matrix, the rows and columns, avoiding the use of individual elements." This is achieved by the following notation: $\mathbf{e}_k$ for the kth unit column vector, $\mathbf{e}_i'$ for the ith unit row vector, $\mathbf{E}_{ki}$ for the matrix $\mathbf{e}_k\mathbf{e}_i'$, $\mathbf{A}_{.k}$ $(=\mathbf{A}\mathbf{e}_k)$ for the kth column of the matrix $\mathbf{A}$, $\mathbf{A}_i$ $(=\mathbf{e}_i'\mathbf{A})$ for its ith row [cf. Nederl. Akad. Wetensch. Proc. Ser. A. **58** (1955), 95–106, 260; MR **17**, 339]. No doubt there are many advantages of this notation (though the number of occurrences in the practical part of the book is not impressive), but it is difficult to see what the author means by saying in his preface that "only with these notations calculation with matrices becomes a 'calculus'."

{This "matrix calculus" is elaborated in Part I (86 pages) of the book. Though it covers the usual subjects of a textbook on matrix theory, it is difficult to use it as an introduction, mainly because of the large number of minor inaccuracies. On one hand the author explains very elementary facts about matrices, in great detail, on the other hand a theorem like Cauchy-Binet's is used without even stating the theorem. The associative law of matrix multiplication is neither stated nor proved, but just used everywhere (in its general form). Several theorems are stated in a simplified form, as if there were no exceptional cases with vanishing determinants or multiple eigenvalues. Neverhteless those who know some matrix theory will find valuable material also in this part I. We especially mention Courant's minimax theorem, and a survey of 20 pages covering theorems on bounds for determinants and eigenvalues.} *N. G. de Bruijn* (Amsterdam).

Block, H. D.; and Rosenbloom, P. C. Perturbations of non-linear eigenvalue problems. Arch. Math. **7** (1956), 172–183.

This paper extends earlier results of Rosenbloom [Arch. Math. **6** (1955), 89–101; MR **16**, 832] on perturbation in Banach space. Because of its complicated nature only a

description is possible. Let X be a Banach space and T a$_0$ (not necessarily linear) mapping of a part D of X into X such that $T_0(x_0)=\lambda_0 x_0$ for x_0 in X and scalar λ_0. Let x_0^* be a bounded linear functional on X such that $x_0^*(x_0)=1$ and $x_0^*(T_0(x)-\lambda_0 x)=0$ for x in D. For each scalar α in a neighborhood B of zero, let $U(\alpha)$ be a map of a neighborhood A of x_0 into X. It is desired to find a solution (x, α) in $A\times B$ of the equations

$$(T_0+U(\alpha))x=(\lambda_0+\alpha)x,\ x_0^*x=1.$$

Under appropriate hypotheses this problem is first reduced to a fix-point problem. With the aid of familiar Lipschitz conditions it is shown that this fix-point problem has a unique solution for sufficiently small perturbations and that this solution may be obtained by iterations. The degree of approximation of the iterants is carefully estimated in terms of the various constants involved. The authors give three examples and remark how their procedure can be modified in several ways. A second method, based on the general implicit function theorem and Newton's method, but requiring assumptions concerning Fréchet differentials is also described.

R. G. Bartle (Urbana, Ill.).

Èskin, L. D. On Euler's algorithm for the extraction of roots. Uč. Zap. Kazan. Univ. **115** (1955), no. 14, 139–143. (Russian)

The author gives a proof of the convergence of an algorithm of Euler for the extraction of integral roots [cf. M. Müller, Math. Z. **51** (1948), 474–496; MR **10**, 574]. If r is positive, and A is a kth order matrix having $a_{ij}=1$ for $i\leq j$ and $a_{ij}=r$ for $i>j$, then $\lambda_0=(r-1)(r^{1/k}-1)^{-1}$ is a simple characteristic value of A whose absolute value is greater than that of any other characteristic value. It is shown that if the column vector X_0 is not orthogonal to the characteristic vector of A belonging to λ_0, the vector $X_n=A^nX_0=(x_1^{(n)},\cdots,x_k^{(n)})$ has the limiting properties $\lim_{n\to\infty} x_i^{(n)}/x_1^{(n)}=r^{(i-1)/k}$ $(i=1, 2, \cdots, k)$. The process is illustrated for $k=3$. The rate of convergence for $k=3$ is estimated by $\rho^{(n)}$ where $\rho=[1+|3r^{1/3}(1+r^{2/3})^{-2}|]^{-1/2}$. The process is also shown to be applicable to complex values of r.

W. S. Loud (Minneapolis, Minn.).

Formica, Gianni. Sull'integrazione approssimata dell'equazione differenziale dei profili di rigurgito delle correnti permanenti gradualmente variate defluenti in alvei cilindrici. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **21(90)** (1956), 78–88.

It is indicated that the Gauss quadrature formula, as well as a modification thereof attributed to Bouzitat, can be applied to a problem in hydraulics in which a definite integral is to be evaluated numerically. The modification utilizes knowledge of the known locations of a zero and a pole of the integrand on the real axis (but outside the interval of integration).

R. D. Richtmyer.

Garbsch, K. Über die Grenzschicht an der Wand eines Trichters mit innerer Wirbel- und Radialströmung. Z. Angew. Math. Mech. Sonderheft (**1956**), S11–S17.

The flow inside a circular cone is considered in the case when the potential flow has a component of velocity about the axis (i.e. a swirling motion) in addition to a component in the radial direction towards the vertex of the cone. Thus, a three-dimensional boundary layer develops along the wall. Since satisfactory solutions of the three partial differential equations for this boundary layer are difficult to obtain by existing methods, the author develops a new and arbitrarily accurate numerical method. This is based on an iteration procedure, in each step of which the solution is required of a pair of ordinary differential equations in a variable η related to the normal distance y from the surface. The distance x from the entrance enters these equations effectively only as a parameter. Detailed numerical results are presented, and the advantages over previous procedures, such as the Pohlhausen approximate technique and the Blasius series method, are discussed. The new method is also applicable to more general rotationally symmetric boundary layers than the one considered.

D. W. Dunn (Baltimore, Md.).

Douglas, Jim, Jr. On the relation between stability and convergence in the numerical solution of linear parabolic and hyperbolic differential equations. J. Soc. Indust. Appl. Math. **4** (1956), 20–37.

Consider the numerical solution of a linear partial differential equation with constant coefficients in the region, D, $0\leq x\leq 1$, $t\leq 0$ for which the solution is specified on the boundaries $x=0$ and $x=1$ and sufficient initial conditions are given to insure the existence of a solution to the problem. Let $(N+1)\Delta x=1$, $x_i=i\Delta x$, $t_n=n\Delta t$, $f_{in}=f(x_i, t_n)$ and $w_{in}=(w_{1n}, w_{2n}, \cdots, w_{Nn})$, where w_{in} is a solution of the difference analogue of the differential equation

(1) $A^{(1)}w_{n+1}=$
$$A^{(0)}w_n+A^{(-1)}w_{n-1}+\cdots+A^{(-q)}w_{n-q}+b_n\ (n\geq q),$$

where $A^{(1)}$ is a non-singular matrix and b_n contains the boundary conditions, among other things. Suppose the matrices $A^{(i)}$ $(i=1, 0, \cdots, -q)$ possess a common set $\{\varphi_p\}$ $(p=1, 2, \cdots, N)$ of eigenfunctions with corresponding eigenvalues $\{\lambda_p^{(i)}\}$. Equation (1) is said to stable if and only if max $|\rho_{pj}|\leq 1$ $(j=1, 2, \cdots, q+1;\ p=1, 2, \cdots, N)$, where the numbers ρ_{pi} are the solutions of the equations

$$\lambda_p^{(1)}\rho^{q+1}-\lambda_p^{(0)}\rho^q\cdots\lambda_p^{(-q)}=0\ (p=1, 2, \cdots, N).$$

The solution u of the differential equation satisfies at interior lattice points the difference equation

$$A^{(1)}u_{n+1}=A^{(0)}u_n+A^{(-1)}u_{n-1}+\cdots+A^{(-q)}u_{n-q}+b_n+h_n,$$

where the h_n are remainders in the Taylor series development of u. Theorem. If $|\lambda_p^{(1)}|\geq K(\Delta t)^\alpha$ $(K>0)$,

$$|\prod_{j\neq k}(\rho_{pj}-\rho_{pk})|\geq L(\Delta t)^\beta$$
$$(L>0;\ j, k=1, 2, \cdots, q+1;\ p=1, 2, \cdots, N),$$

(1) is stable, $\|u_i-w_i\|_{l_2}=O((\Delta t)^{r+\beta}$ $(i=0, 1, \cdots, q)$, and $\|h_n\|_{l_2}=O((\Delta t)^{s+\alpha+\beta+1}$ $(n\geq q)$, then the solution of (1) converges uniformly in D in the l_2 topology to the solution of the corresponding differential equation and $\|u_n-w_n\|_{l_2}=O((\Delta t)^r+(\Delta t)^s)$. Moreover, by extending w to all of D by multilinear interpolation, the author also obtains convergence in the mean in the L_2 topology. Applications of this theorem to hyperbolic and parabolic equations are given.

J. K. Hale (Albuquerque, N.M.).

Wasow, Wolfgang. Discrete approximations to elliptic differential equations. Z. Angew. Math. Phys. **6** (1955), 81–97.

This paper studies the accuracy of discrete approximations to solutions of boundary value problems for elliptic equations in a rather general framework introduced by Petrowsky. A measure for the accuracy of an approximation scheme in the interior and near the boundary is introduced, in terms of which the truncation

error is estimated. This estimate is then applied to a number of schemes in current use. The size of the error depends crucially on the treatment of points near the boundary of the domain. In particular, the error is a differentiable function of the meshwidth h at $h=0$ only if the scheme near the boundary is accurate enough. This limits the applicability of Richardson's extrapolation method. *P. D. Lax* (New York, N.Y.).

Ghermănescu, M. Finite difference equations. Gaz. Mat. Fiz. Ser. A. 8 (1956), 282–296. (Romanian)

An expository paper on the theory of equations with finite differences and their application to the approximate solution of ordinary and partial differential equations. *E. Grosswald* (Philadelphia, Pa.).

Zierep, Jürgen. Über ein Charakteristikenverfahren zur angenäherten Berechnung der unsymmetrischen Überschallströmung um mehrere hintereinander angeordnete ringförmige Körper. Z. Flugwiss. 4 (1956), 290–300.

The method of characteristics has been applied to linearized supersonic flow problems by Haack [Z. Angew. Math. Phys. 2 (1951), 357–375; MR 13, 597] and by Erdmann and Oswatitsch [Z. Flugwiss. 2 (1954), 201–215]. It is useful for obtaining numerical solutions of small-perturbation flows past inclined bodies of revolution. Here the work of Haack on ring-shaped wings at incidence is extended to consideration of the regions affected by the trailing edges. Since the ring-wings carry lift, there are ring-shaped vortex sheets behind them. The conditions of flow at such a sheet are determined here by requiring continuity of flow direction and pressure. Together with initial values along the Mach lines springing from the trailing edge, these conditions determine the flow. This method is applied to cases where one ring-wing is placed behind another, both encircling a body of revolution. Numerical examples are given. *W. R. Sears.*

Hall, M. G. The accuracy of the method for characteristics for plane supersonic flow. Quart. J. Mech. Appl. Math. 9 (1956), 320–333.

This is a study of the propagation of errors in the step-by-step computations by the method of characteristics. The particular case studied is isentropic, steady, plane flow. The equations of propagation of error are set up and are solved approximately for small errors. The solutions show how the error is propagated along the characteristics through the point where it is introduced and also in the region between them. Various kinds of errors are considered, such as mean-slope errors, which are due to the use of mean values of the slopes of characteristics in each step and are therefore systematic, and rounding-off errors, which are random. The paper closes with a note on the planning of computations. *W. R. Sears.*

Lemaitre, G. Intégration par analyse harmonique. Ann. Soc. Sci. Bruxelles. Sér. I. 70 (1956), 117–123.

The author considers the problem of obtaining approximate numerical values of the solutions of differential equations of the form

$$L(y)=z(x, y), \tag{1}$$

where $L(y)$ is linear in $y=y(x)$ and its derivatives and $z(u, v)$ is a given function of u and v. It is further supposed that both y and z, considered as functions of x, are periodic of period 2π and have a mean value of zero on this interval.

The approximating trigonometric polynomials

$$y(x)\sim\sum_{q=1}^{k}(a_q\cos qx+b_q\sin qx)$$

$$z(x)\sim\sum_{q=1}^{k}(A_q\cos qx+B_q\sin qx)$$

have coefficients $a_qb_qA_qB_q$ which would be related via (1) and so there would be the possibility of determining the a_q and b_q from the given information. However the author proposes to eliminate these coefficients in favor of relationships between values of y and z for equally spaced values of x in the interval $(0-2\pi)$, or indeed any interval of this length. Supposing that $x_j=j\pi/k$, $j=-k(1)k-1$, and setting $y_j=y(x_j)$, $z_j=z(x_j)$, we have by inversion

$$kA_q=\sum_{j=1}^{k}z_j\cos qx_j,\quad kB_q=\sum_{j=1}^{k}z_j\sin qx_j.$$

From these relations and (1) we deduce that

$$y_r=y(x_r)=\sum_{j=1}^{k}L_jz(x_j+x_r), \tag{2}$$

where the coefficients L_j depend only on k and the operator L. For the case $L(y)=dy/dx$ we have

$$kL_j=-\sum_{q=1}^{k}\frac{\sin qx_j}{q}$$

and for $L(y)=d^2y/dx^2$ we have

$$kL_j=-\sum_{q=1}^{k-1}\frac{\cos qx_j}{q^2}-\frac{\cos kx_j}{2k^2}$$

Both sets of coefficients are tabulated to 6D for $k=4$, 8 and 16. The right member of (2) contains y_j implicitly and so the problem is reduced to solving for the y's by iteration. The process is illustrated numerically by solving the equation

$$\frac{d^2y}{dx^2}=-4\sin y\ \ (y(0)=0).$$

Results are compared with the known solution in terms of elliptic functions. The agreement is good to about 4 significant decimal places. *D. H. Lehmer.*

See also: Fan, p. 183; Head, p. 207; Boley, p. 216; Tompkins, p. 238; Binet and Watson, p. 243; Press and Tukey, p. 245; Karas, p. 247; Pólya, p. 250; Häfele, p. 254; Owczarek, p. 256; Rubinaw and Wu, p. 257; Risberg, p. 259; Šaraf, p. 263; Thompson, p. 264.

Graphical Methods, Nomography

★**Kiessler, Fritz. Nomographisches Rechnen.** Verlag W. Girardet, Essen, 1956. 190 pp. DM 9.80.

The book, intended for student and engineer, has three parts: two variables, three variables, more than three variables. There are many exercises and illustrations but little theoretical discussion. Very little knowledge of mathematics is presupposed.

★**Morita, Katuhiko. Nomography.** Tôkai-shobô, Tokyo, Japan. 170 pp. 160 yen. (Japanese)

The book has ten chapters with the (abbreviated) headings: Concept of nomography, charts of two vari-

ables, coordinate systems, determinants, network and alignment charts, charts of three variables (chapters 6, 7, 8), charts of more than three variables, movable scale charts. The main features of the book are that the determinantal method is adopted almost throughout and that network and alignment charts are directly compared on the same page after the manner of dual theorems in projective geometry. *From the author's summary.*

See also: Kleeman, p. 250; Volt, p. 256.

Tables

Teichroew, D. Tables of expected values of order statistics and products of order statistics for samples of size twenty and less from the normal distribution. Ann. Math. Statist. 27 (1956), 410–426.

Let $x_1, x_2, \cdots, x_N$ be a sample from the normal distribution with mean 0 and variance 1, arranged in order of size so that $x_1 \geqq x_2 \geqq \cdots \geqq x_N$. The means, variance and covariances of these "order statistics" may be obtained from $E(x_i)$, $E(x_i x_j)$. The computation of these involved the use of

$$\psi(a)=\int_{-\infty}^{\infty}[F(x)]^a[1-F(x)]^a dx,$$

$$\psi(a, b)=\int_{-\infty}^{\infty}[F(x)]^a dx \int_x^{\infty}[1-F(y)]^b dy=\psi(b, a).$$

Tables of $E(x_i)$ and $E(x_i x_j)$ are given rounded to ten decimal places for $N<20$. Tables of $\psi(a)$ and $\psi(a, b)$ are given rounded to 25 decimal places. In the table of $\psi(a)$, $1\leqq a\leqq 10$ and in that of $\psi(a, b)$, $1\leqq a\leqq b\leqq 19$. The method of computing and checks are discussed. It is pointed out that the tables of $E(x_i)$ and $E(x_i x_j)$ are available to twenty decimal places. *H. Chernoff* (Stanford, Calif.).

Sarhan, A. E.; and Greenberg, B. G. Estimation of location and scale parameters by order statistics from singly and doubly censored samples. Ann. Math. Statist. 27 (1956), 427–451.

This paper gives best linear systematic estimates of the mean and standard deviation of a normal distribution from singly and doubly censored samples of size $\leqq 10$. The following useful tables are in the paper: (I) Variances and covariances of order statistics in samples of size up to 20 from a normal population. The entries are given to ten decimal places. These tables were made possible by the existence of the tables reviewed above; (II) Coefficients of the most efficient linear systematic estimates of the mean and standard deviation in censored samples of size 10 from a normal distribution; (III) Variances and covariances of the estimates of the mean and standard deviation for censored samples up to size 10 from a normal distribution; (IV) Percentage efficiencies of the estimates of the mean and standard deviation for censored samples relative to uncensored samples up to size 10 from a normal distribution. (V) Variances and relative efficiencies of alternative estimates of the mean and standard deviation for censored samples of size 10 from a normal distribution. The alternative estimates are obtained by an extension of results of A. K. Gupta [Biometrika 39 (1952), 260–273; MR 14, 487]. In finding the relative efficiency one compares the variance of the alternative estimate with the variance of the corresponding best linear systematic estimate. *B. Epstein.*

Douglas, J. B. Tables of Poisson power moments. Biometrika 43 (1956), 489.

Mikić, Fedor. Table to calculate the coefficients in the logarithmic curve for solving the trend and other stochastical relations. Jugoslav. Akad. Znan. Umjet. Rasprave Odj. Mat. Fiz. Tehn. Nauke 1 (1952), no. 1, 11 pp. (1 insert). (Serbo-Croatian. English summary)

Coefficients of the logarithmic trend $y=a+bx+c \log x$ are computed by solving the normal equations. Many calculations needed for establishing these coefficients are reduced to less than one third by aggregates A, B, C, γ, E, F of basic sums tabulated in the appendix for $x=1, 2, \cdots, 100$. Examples of numerical computation schemes for the regression curves are given. *J. Janko.*

See also: Rathie, p. 208; Lemaitre, p. 237; Barton and David, p. 240; Kleeman, p. 250; Šaraf, p. 263; Jung, p. 264.

Machines and Modelling

★**Tompkins, C. Machine attacks on problems whose variables are permutations.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 195–211. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

The author discusses a number of problems involving exhaustive search over sets of permutations from the point of view of the high speed digital computer.

The problem roughly consists of describing what happens when a very fast computer, like the SWAC, comes in contact with simple functions of a very large number, $n!$, of variables. Problems of this type include the search for latin squares, the traveling salesman problem, the assignment problem, and the discovery of various algebraic systems.

Basic to all these permutation problems is that of the economical and systematic generation of permutations on more than a few letters. The author describes two such methods. One method generates the permutations in lexicographical order and is conveniently interrupted and recommenced elsewhere as needed in problems in which large blocks of permutations are rejected for some reason. The second method assigns a signature to a permutation from which the permutation can be generated. Signatures can then be generated systematically or at random. This enables the generation of "random permutations" useful in applying the Monte Carlo methods to the sampling of permutations too numerous to deal with exhaustively.

The author considers various attempts to embed permutation problems in continuous spaces so that recourse can then be had to gradient methods as for example the introduction of matrices whose elements lie between 0 and 1 as a continuous generalization of the permutation matrix. In most cases this has not led to really successful machine methods since the analysis of the generalized problem is plagued by numerous local maxima. The article concludes with a section on the problem of economic computing.

The paper is a fine source of names of workers in this field but much of the work ascribed to these workers is as yet unpublished. There is a bibliography of ten titles. *D. H. Lehmer* (Berkeley, Calif.).

★**Lytle, Ernest J., Jr. A description of the generation and testing of a set of random normal deviates.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 234–248. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

This is an account of a study of 25000 random normal deviates generated from a rectangularly distributed set of random numbers drawn from a set produced by RAND. More precisely random values were attributed to the integral

$$(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{X}\exp(-\tfrac{1}{2}t^2)dt$$

and the corresponding values of X were computed to 5D. The resulting set having mean value $-.00412$ and variance $.99184$, instead of the expected values 0 and 1, was subjected to several tests. Except for the fact that there were too many occurrences of duplicate extremely rare values, a phenomenon the author claims is due to the method of determining X, the author regards the results as satisfactory. *D. H. Lehmer* (Berkeley, Calif.).

★**Taussky, Olga; and Todd, John. Generation and testing of pseudo-random numbers.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 15–28. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $ 7.50.

This paper is a report on the literature of pseudo-random numbers together with the results of tests applied to a certain "additive" method described below. The report stresses uniform, or rectangular, distributions (rather than normal or binomial distributions) of random numbers (rather than random digits). Emphasis is on speed of production by automatic digital computers as well as quality of the product.

Four methods are discussed. (a) Mid-square methods of Von Neumann and Metropolis. (b) Multiplicative congruential methods of Lehmer and Taussky-Todd. (c) Additive congruential methods of Duparc Lekkerkerker and Peremans. (d) Periodic patterns method of Rosser.

In method (a) the central n digits are extracted from the square of the n digit random number X_k to produce X_{k+1}. Precautions are needed to avoid degeneration to $X_k=0$. In certain cases the method has been satisfactory.

In (b) the sequence X_k is defined recursively by

$$X_{k+1}=\rho X_k \pmod{M}.$$

Here the ρ is chosen to produce a very long period such as a primitive root when M is a prime and a power of 5 when M is a power of 2. To avoid actual division M can be either a power of 2 or 10 (depending on the type of machine) or a few units more or less than such a power. In the first case only the most significant digits of X_k are random and the X_k are to be regarded as approximate real numbers, say between 0 and 1. Tests of 16384 numbers thus generated with $M=2^{42}$ are described. These are for (1) frequency, (2) distribution of certain moments, (3) runs up and down and above and below the mean. All tests were passed satisfactorily.

In (c) the attempt is made to cheapen the cost of production by replacing multiplication by addition. Here the sequence considered was defined by the Fibonacci congruence

$$X_{k+1}\equiv X_k+X_{k-1} \pmod{2^{44}}$$

which has a period of 3.2^{43} and is obtained very cheaply by simply adding without regard for overflow.

The authors regard the method as satisfactory for tests (1) and (2) but not for (3), there being a preponderance of runs of length 2. They point out that since the Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, $\cdots$ are approximately proportional to the successive powers of a quadratic irrational we might expect a behaviour similar to that of method (b).

In (d) the machine is made to produce four periodic patterns of lengths 31, 33, 34 and 35. The resulting four infinite binary numbers are then added without carry to produce a random sequence of binary digits. Tests by Forsythe on sums of 100 consecutive digits gave satisfactory results in eleven out of twelve cases.
D. H. Lehmer (Berkeley, Calif.).

Jaspen, Nathan. Machine computation of higher moments. J. Amer. Statist. Assoc. **51** (1956), 489–500.

An efficient method is presented for the economical and rapid computation of as many as six to twelve moments from a single run of the tabulating cards through the IBM Accounting Machine Type 402. The only auxiliary machine required is a sorter. Three features of the accounting machine are employed: progressive summation, total transfer, and the special program device. Moments are available from the successive progressive summations of either the scores or the card count, and both techniques can be applied simultaneously to provide a check on the computations. (Author's summary.)
H. Chernoff (Stanford, Calif.).

Pinsker, I. Š. Representation of functions of several variables by means of adders, multipliers and simple functional instruments. Trudy Inst. Mašinoved. Sem. Točn. Mašinostro. Priborostr. **8** (1955), 35–51. (Russian)

Functions of one or two independent variables can be simply and precisely mechanized. Processes of addition and multiplication may also be simply mechanized. Author proposes a scheme for simple mechanization of a function of n variables by approximating it with sums and products of functions which individually contain not more than two of the n variables. Thus, $f(x, y, t)$ may be represented by sums of terms of the following types:

$$\alpha(x)\cdot\beta(y)\cdot\tau(t);\ \alpha(x,t)\cdot\beta(y,t)\cdot\tau(x,y);\ \alpha(x)\cdot\beta(y,t);$$

$\alpha(x,y)\cdot\beta(x,t)$; etc. Only enough terms are selected to give the desired precision, and this is done by successive compensation to minimize the number of terms. A numerical example converts a set of tabulated values for $f(x,y,t)$ to the form $\beta_1(y,t)+\alpha_2(x)\cdot\beta_2(y,t)$ with 5-place precision.
W. W. Soroka (Berkeley, Calif.).

Burks, Arthur W.; and Copi, Irving M. The logical design of an idealized general-purpose computer. I, II. J. Franklin Inst. **261** (1956), 299–314, 421–436.

The authors present the complete logical design of an idealized general-purpose digital computer. The computer is a sixteen bit, two's complement, parallel machine with 4096 random access bins of storage, the last of which contains the input or output word to an external serial storage (e.g. a doubly infinite tape). The four bit order code includes eight operational instructions (clear add, clear subtract, hold add, hold subtract, double, halve, store, and extract address), four control instructions (halt, jump, jump on negative, and jump on odd), and two editing instructions (forward one, and back one). Multiplication, division, general shifting, general extracting, and sensing for overflow must be programmed.

The logical elements used are the delay element, and the stroke function. For simplicity the derived functions, negation, conjunction, disjunction, implication, equivalence, and inequivalence, are added to the logical building blocks.

In addition to the main block chart, the authors give the logical details of the serial and parallel storage, the arithmetic unit, and, of the control, the address decoder, the address counter, the operand decoder, and the control clock.

This paper has been found useful in the introductory course to digital computers given by the Moore School of Electrical Engineering. *S. Gorn* (Philadelphia, Pa.).

Patterson, George W. Symbolic methods in the design of delay- and cycle-free logical nets. Convention Record of the I. R. E. 2 (1954), part 4, 58–64.

The author describes the relationship between the logical design of machines and the propositional calculus via truth tables. The use of the calculus in the analysis, synthesis, and optimization of logical nets is described in several examples. *S. Gorn* (Philadelphia, Pa.).

See also: Hall and Swift, p. 192; Cohn, p. 194; Šaraf, p. 263.

PROBABILITY

Pacheco de Amorin, Doutor Diogo. Calculus of probability and classification of sciences. Las Ciencias 21 (1956), 365–381. (Portuguese)

Vietoris, L. Häufigkeit und Wahrscheinlichkeit. Studium Gen. 9 (1956), 85–96.

The author presents a system of axioms for the foundations of the theory of probability. The system is somewhat similar to that of Koopman [Ann. of Math. (2) 41 (1940), 269–292; MR 1, 245] and is based on a linear ordering of events in which order is interpreted in terms of likelihood. The axioms are natural and produce numerical probabilities as follows: For each positive integer n the system is required to contain a set of n disjoint, exhaustive and equally likely events. Unions of these events are events whose probabilities are rational fractions with denominator n. The ordering then assigns to an arbitrary event a probability which is a Dedekind cut among the rationals. In applying the system, equal likelihood is defined in terms of symmetry with respect to certain properties which are described as primary—shape, mass distribution, etc. With respect to the relation between axiom systems and their applications the author takes sharp issue with Carnap [Logical foundations of probability, Univ. of Chicago Press, 1950; MR 12, 664] and Stegmüller [Bemerkung zum Wahrscheinlichkeit — problem Stud. Gen. 6 (1953) pp. 563–593]. The author also criticizes the frequency theory of probability and points out that the limiting frequency can have no bearing on any finite (and hence any observable) frequency. He states that Bernoulli's theorem contains the important relation between probabilities and frequencies.

A. H. Copeland (Ann Arbor, Mich.).

Barton, D. E.; and David, F. N. Some notes on ordered random intervals. J. Roy. Statist. Soc. Ser. B. 18 (1956), 79–94.

The random intervals are those obtained by dropping a given number of points randomly on a line of unit length; they are ordered by size. Joint probability densities for one or more of these ordered intervals are obtained in the form of multiple sums and the first few moments and cross products for singles, pairs and triples are evaluated explicitly; also multivariable ordinary moment generating functions, of the symbolic form $(1-m_1t_1-m_2t_2-\cdots)^{-n}$, $m_1^{k_1}m_2^{k_2}\cdots\equiv m_{k_1,k_2,\ldots}$, with n the number of intervals, are determined. Some limiting distributions are given. Finally, the distributions of the ratio and difference of two ordered intervals are derived. Eight tables of numerical results appear. *J. Riordan.*

Breny, H. A propos de la notion de fonction aléatoire. Ann. Soc. Sci. Bruxelles. Sér. I. 70 (1956), 96–116.

Expository paper. *J. L. Doob* (Geneva).

Hoeffding, Wassily. On the distribution of the number of successes in independent trials. Ann. Math. Statist. 27 (1956), 713–721.

Let $X_1, \cdots, X_n$ be independent with $P\{X_i=1\}=1-P\{X_i=0\}=p_i$ and let $S=\sum_1^n X_i$. The author proves, subject to the restriction $\sum p_i=np$, (a) that $Eg(S)$ is maximized by a set of p_i taking on three values of which two are 0 and 1, (b) that all $p_i=p$ gives the unique maximizer if g is strictly convex, and (c) that, for $b\leqq np\leqq c$, $P\{S\leqq b\}$ is maximized and $P\{S\leqq c\}$ and $P\{b\leqq S\leqq c\}$ are minimized when all $p_i=p$. (b) and (c) can be used in certain estimation problems to reduce considerations to the "worst case" where all $p_i=p$. *J. Kiefer.*

Geisser, Seymour. A note on the normal distribution. Ann. Math. Statist. 27 (1956), 858–859.

Consider a sample of n independently and identically distributed observations and suppose that the corresponding distribution has finite variance. Denote by $\bar{x}$ the sample mean and by δ_k the mean square successive difference with lag k. The author characterizes the normal population by the property that $\bar{x}$ and δ_k are stochastically independent. This theorem is a particular case of a more general result of the reviewer [Proc. 3rd Berkeley Symposium Math. Statist. Probability, 1954, 1955, v. 2, Univ. of California Press, 1956, pp. 195–214] which however was not known to the author at the time his note was written. *E. Lukacs* (Washington, D.C.).

Darling, D. A. The maximum of sums of stable random variables. Trans. Amer. Math. Soc. 83 (1956), 164–169.

Let $X_1, X_2, \cdots$ be identically distributed independent random variables and let $S_n=X_1+\cdots+X_n$. Using a result due to Spitzer [same Trans. 82 (1956), 323–339; MR 18, 156] the author finds the limiting distribution of the one-sided maximum $\max_{1\leqq k\leqq n} S_k$ when the X_i have a symmetric stable distribution of index γ, $0<\gamma\leqq 2$. When the X_i are normally distributed, Bachelier [Les lois des grands nombres du calcul des probabilités, Gauthier-Villars, Paris, 1937] gave the limiting distribution of $\max S_k$ as the truncated normal distribution and the limiting distribution of $\max |S_k|$ as a theta function. Erdös and Kac [Bull. Amer. Math. Soc. 52 (1946), 292–302; MR 7, 459] showed these results also hold when the X_i are not necessarily identically distributed but do obey the central limit law. Kac and Pollard [Canad. J. Math.

2 (1950), 375–384; MR 12, 114] found the limiting distribution of $\max|S_k|$ when the X_i are identically distributed Cauchy random variables. The limiting distribution of $\max|S_k|$ for general stable random variables remains apparently undetermined.

H. P. Emundson (Pacific Palisades, Calif.).

Alda, Václav. On conditional expectations. Czechoslovak Math. J. **5(80)** (1955), 503–505. (Russian summary)

Let $F_1 \subset F_2 \subset \cdots$ be a monotone sequence of Borel fields of measurable sets of a probability measure space, and let f be a function measurable with respect to the least Borel field containing every F_j. Then, if f has a finite integral, the conditional expectation of f with respect to F_n converges to f almost everywhere. The author (who phrases the theorem unnecessarily restrictively in terms of coordinate functions on product spaces) gives a simple direct proof of this known theorem. The theorem goes back, in a somewhat simpler form, to Lévy [Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris, 1937]. *J. L. Doob* (Geneva).

Korolyuk, V. S. Asymptotic expansions for distributions of maximum deviations in the scheme of Bernoulli. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 183–186. (Russian)

Let $\xi_1, \cdots, \xi_n$ be mutually independent random variables with a common distribution function. Each random variable takes on only two values, chosen so that it has a non-negative median, mean 0, variance $1/n$. Let $S_0=0$, and otherwise $S_k=\sum_1^k \xi_r$. The author sketches the derivations of asymptotic expressions, with remainder $O(1/n)$, for the probabilities $P\{\max_{0\leq r\leq n} S_r<z\}$ and $P\{\max_{0\leq r\leq nt} S_r<z|S_{nt}=x\}$, as well as these probabilities with S_r replaced by $|S_r|$. As an application, if $\{\zeta_\lambda(t), t\geqq 0\}$ is the Poisson process, with parameter λ,

$$P\{\max_{0\leq t\leq 1}[{}_{0\leq t\leq 1}-\zeta_\lambda(t)/\lambda]<z\},$$

and the same expression with the brackets replaced by absolute value signs, are evaluated as $\lambda\to\infty$ up to terms of order $1/\lambda$. [See also Chung, Trans. Amer. Math. Soc. **64** (1948), 205–233; MR **10**, 132.] In an earlier paper of the author [Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 103–124; MR **16**, 839] the special case of symmetric distributions of the summands was treated.

J. L. Doob (Geneva).

Jensen, Arne. Application of stochastic processes to an investment plan. Metroecon. **5** (1953), 129–137.

Using a Markov process to describe the demand for telephone traffic in a given region, and assuming that money is invested in equipment whenever the demand exceeds the capacity of the telephone plant to provide a given grade of service, the author describes a method of calculating the present value of an investment. The presentation is brief, and details are promised in future work.

V. E. Beneš (Murray Hill, N.J.).

★**Rosenblatt, Murray. Some purely deterministic processes.** Research Division, College of Engineering, New York University, New York, N.Y., 1956. 21 pp.

The author considers two spectral densities for purely deterministic stationary discrete parameter stochastic processes, and, in each case, finds the rate of approach to 0 of the mean square prediction error for a knowledge not of the complete past but only of the preceding n instants. In one case, for example, the spectral density is continuous and positive on $[\frac{1}{2}\pi-\alpha, \frac{1}{2}\pi+\alpha]$ and vanishes elsewhere on $[0, 2\pi]$. In this case the above prediction error has the order of magnitude $(\sin\frac{1}{2}\alpha)^{2n+1}$. *J. L. Doob* (Geneva).

Palmer, D. S. Properties of random functions. Proc. Cambridge Philos. Soc. **52** (1956), 672–686.

The author considers two stationary, normally distributed random variables $f(t)$, $g(t)$ of zero mean. He defines the concentration of their zeros as $X=\lim\{(T/2h)$ (number of zeros of f within a distance h of a zero of g in $0<t<T)/$(number of zeros of f)(number of zeros of g)$\}$, where $T\to\infty$, $h\to 0$, $T^2h\to\infty$. By arguments patterned on those of Rice [Bell System. Tech. J. **24** (1945), 46–156; MR **6**, 233], but without stating conditions involved, he calculates an explicit formula for X in terms of the values of $P(0)$, $P''(0)$, $Q(0)$, $Q''(0)$, $R(0)$, $R'(0)$, $R''(0)$, where $P(x)=E(f(t)f(t+x))$, $Q(x)=E(g(t)g(t+x))$, $R(x)=E(f(t)g(t+x))$. For particular choices of the function P, Q, R, the author compares his calculated X with some previously published experimental data [e.g., Briggs and Page, The physics of the ionosphere, Physical Society, London, 1955, pp. 119–122]. The paper also considers concentration of extremal points and distributions of intervals between zeros or extremal values.

P. Hartman (Baltimore, Md.).

Hájek, Jaroslav. Linear estimation of the mean value of a stationary random process with convex correlation function. Czechoslovak Math. J. **6(81)** (1956), 94–117. (Russian. English summary)

Let $x(t)$ be a stationary process with correlation function

$$R(\tau)=\frac{Ex(t)x(t+\tau)-[Ex(t)]^2}{Ex^2(t)-E[x(t)]^2}.$$

A linear estimate for $Ex(t)$ over the interval $(0, T)$ is $\bar{x}=\int_0^T x(t)d\Phi(t)$, where $\Phi(t)$ is a function of bounded variation with $\Phi(0)=0$ and $\Phi(T)=1$. The main result of the paper is to give a lower bound for the variance of such an estimate in the case that $R(\tau)$ is a convex (downward) function. Specifically it is proved in this case that

$$\liminf_{T\to\infty} T\,D^2\Big[\int_0^T x(t)d\Phi_T(t)\Big]\geqq 2\sigma^2\int_0^\infty R(\tau)d\tau.$$

The author also obtains results relating to the best estimate (minimum variance) over a fixed interval. Several examples are given. *J. L. Snell.*

Champernowne, D. G. An elementary method of solution of the queueing problem with a single server and constant parameters. J. Roy. Statist. Soc. Ser. B. **18** (1956), 125–128.

Elementary derivation of one of the results of Clarke [Ann. Math. Statist. **27** (1956), 452–459; MR **18**, 157].

J. Wolfowitz (Ithaca, N.Y.).

Lah, Ivo. Analytical graduation of fertility rates. J. Amer. Statist. Assoc. **51** (1956), 461–466.

Sugiyama, Hiroshi; and Kitabatake, Satoshi. On a method of truncated life-testing. Math. Japon. **3** (1955), 152–160.

The authors consider truncated life tests in a two state, two action situation where the action taken depends on how long it takes for the rth smallest value $x_{r,n}$ in a random sample of size n to occur. In the special cases

where life times are normally or exponentially distributed, they discuss how one chooses r, n, and the truncation time so as to meet a prescribed α, β condition and so as to minimize a cost C of the form c_1n+c_2n, where c_1 and c_2 are prescribed positive constants. {Unfortunately the cost used by the authors does not include a waiting time term, which is often the most important quantity in life test problems.} *B. Epstein* (Detroit, Mich.).

Schmetterer, Leopold. Die Risikotheorie in der Versicherungsmathematik. Statist. Vierteljschr. 9 (1956), 1–15.

Ridderström, Sven. On ratio estimates in simple random sampling with some practical applications. Skand. Aktuarietidskr. 38 (1955), 135–162.

When estimating by random sampling the fraction between two statistical variables one may use the fraction between the two means and express the variance by the variances of the two variables and the correlation coefficient as well.

The theorems deduced are used for estimating various data for life assurance valuation. The distribution of the variances for subgroups is compared with the normal distribution. *P. Johansen* (Copenhagen).

Fortet, Robert. Les fonctions aléatoires en téléphonie automatique. Probabilités de perte en sélection conjuguée. Ann. Télécommun. 11 (1956), 85–88.

This is an expository article of mathematical methods applicable to busy signal telephone systems, those in which calls arriving when the system is fully occupied are refused and constitute the loss. The first part formulates an integral identity of Stieltjes type for the number of elements (trunks, switches, servers) occupied at a given time in terms of the arrival density and characteristic functions of the service time and the time of full occupancy. This equation is more fully considered elsewhere by the author [Proc. 3rd Berkeley Symposium Math. Statist. Probability, 1954, 1955, v. 2, Univ. of California Press, 1956, pp. 81–88]. The second part considers loss calculations in "link systems" and is a summary of a paper presented to the first international traffic congress held in Copenhagen in June of last year. *J. Riordan.*

★**Lloyd, S. P.; and McMillan, B. Linear least squares filtering and prediction of sampled signals.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 221–247. Polytechnic Institute of Brooklyn, Brooklyn, N.Y., 1956.

Let $s(t)$ (the "signal"), and $r(t)$ (the "noise") be stationary in the wide sense, and τ, $T>0$, fixed parameters. The problem is to find a linear operation ϕ on the samples $r(m\tau)$, $m=0, \pm1, \pm2, \cdots$, $m\tau<t$, such that the "error", $\xi(t)=\phi(t)-s(t-T)$, is minimized in the sense of minimizing

$$P(K)=\tau^{-1}\int_0^\tau E[\xi^2(t)]dt.$$

Particular attention is paid to operators of the form (*) $\phi(t)=\tau\sum_m r(m\tau)K(t-m\tau)$, where $K(t)=0$, $t<0$. The methods used are, in part, adapted from Chapter 12 of Doob's "Stochastic processes" [Wiley, New York, 1953; MR 15, 445]. When $n(t)$ is a process of moving averages, Wiener-Hopf factorization methods apply. The results are most explicit with rational spectral densities, e.g. exponential autocorrelations.

In section 7, there is a discussion of realizing the operator (*) by a physical network. *E. Reich.*

See also: Bopp, p. 259.

STATISTICS

Hudimoto, Hirosi. Note on fitting a straight line when both variables are subject to error and some applications. Ann. Inst. Statist. Math., Tokyo 7 (1956), 159–167.

The author uses a method of Wald [Ann. Math. Statist. 11 (1940), 285–300; MR 2, 108] to fit the straight line $Y=\alpha+\beta X$ to cement strength data, where X and Y are measurements separated by several days. A procedure is also presented for estimating the probability (p) that a subsequent observed Y will exceed its expected value, and approximate confidence limits are given for p.

R. L. Anderson (Raleigh, N.C.).

Landenna, Giampiero. La dissomiglianza. Statistica, Bologna 16 (1956), 21–57.

Let X and Y be two random variables with distribution functions $F_1(x)$ and $F_2(y)$ respectively. Fréchet [Ann. Univ. Lyon. Sect. A (3) 14 (1951), 53–77; MR 14, 189] studied the family Φ of bivariate distributions $F(x, y)$ which have $F_1(x)$ and $F_2(y)$ as marginal distributions. The present author uses Fréchet's results to define measures of association between the random variables X and Y. The "index of dissimilarity" $d_{XY}^{(r)}$ of order r is defined as

$$d_{XY}^{(r)}=\min\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x-y|^r dF(x, y)\right\}^{1/r},$$

where the minimum is to be taken over all $F(x, y)\in\Phi$. In the first part of the paper these indices are studied, the second part deals with the corresponding sample characteristic for $r=1$. *E. Lukacs* (Washington, D.C.).

Janne d'Othée, Henry. Sur les polynômes employés dans les régressions de divers degrés en Statistique. Bull. Soc. Roy. Sci. Liège 25 (1956), 251–259.

An expository article showing how to determine constants A_p, $a_{p,i}$ ($i=0, 1, \cdots, p$; $p=0, 1, \cdots, n$) with $a_{p,p}=1$, so that

$$y=\sum_{p=0}^{n} A_p\left(\sum_{i=0}^{p} a_{p,i}x^i\right)$$

passes through the points $(0, y_1)$, $(1, y_2)$, $\cdots$, $(n-1, y_n)$, where the polynomials $\sum_{i=0}^{p} a_{p,i}x^i$ ($p=0, 1, \cdots, n$) are orthogonal over the set of values $x=0, 1, \cdots, n-1$.

S. S. Wilks (Princeton, N.J.).

Fabian, Václav. A decision function. Czechoslovak Math. J. 6(81) (1956), 31–45. (Russian summary)

Let $X_1, X_2, \cdots$, be independent and identically distributed random variables with $P(X_i=1)=p$ and $P(X_i)=0)=1-p$. It is assumed that p is an element of the set $\mathfrak{M}=\langle 0, \frac{1}{2})\cup(\frac{1}{2}, 1\rangle$. A sequential decision procedure is given to choose between the alternatives $p<\frac{1}{2}$ and $p>\frac{1}{2}$ satisfying (a) the probability of an incorrect decision is $\leq\alpha$ for all $p\in\mathfrak{M}$ and (b) with probability one (for each $p\in\mathfrak{M}$) the procedure will choose a decision after a finite number of observations. An upper bound for the median of the number of observations required is given. No optimal properties are claimed for the test. *J. L. Snell.*

Fraser, D. A. S. Sufficient statistics with nuisance parameters. Ann. Math. Statist. 27 (1956), 838–842.

Consider a family of distributions $\{P_\theta, \eta\}$ with the pair of indices (θ, η) varying over a Cartesian product $\Theta \times H$. The author defines a statistic T to be sufficient for θ if the marginal distribution of T and the conditional distribution given T are independent of η and θ respectively. He shows that this type of statistic retains some of the properties of sufficient statistics in connection with estimation and hypothesis testing. *E. L. Lehmann.*

Bartholomew, D. J. A sequential test for randomness of intervals. J. Roy. Statist. Soc. Ser. B. 18 (1956), 95–103.

P. A. P. Moran's test [same J. 13 (1951), 147–150; MR 13, 667] for the hypothesis that a given sequence of n intervals is derived from a Poisson process (independent intervals with a common exponential distribution) against the alternative that the common distribution is chi-square with $2a$ degrees of freedom is given a sequential form, first assuming the mean number of occurrences of events known, and then that this mean is unknown. For the first case, both the operating characteristic and the average sample number may be given in relatively simple form. For the second, it is shown that the operating characteristic for all practical purposes is the same, but that the average sample number may be appreciably greater. *J. Riordan* (New York, N.Y.).

Stein, Charles. The admissibility of Hotelling's T^2-test. Ann. Math. Statist. 27 (1956), 616–623.

Extending some work of Birnbaum [same Ann. 26 (1955), 21–36; MR 16, 729], the author obtains a new sufficient condition for the test of a hypothesis concerning an exponential family of distributions to be admissible against unrestricted alternatives. This is applied to prove admissibility of Hotelling's T^2-test. *E. L. Lehmann.*

Hájek, Jaroslav. The asymptotic efficiency of a certain sequence of tests. Czechoslovak Math. J. 6(81) (1956), 26–30. (Russian. English summary)

Let $X_1, \cdots, X_n$ be n identically and independently distributed chance variables and $k<n$. The α_k-test is defined by $\alpha_k = \sum_{1\leq i_1<\cdots<i_k\leq n} \varphi(X_{i_1}, \cdots, X_{i_k})$, where $\varphi(x_1, \cdots, x_k)=1$ or 0 depending on whether $x_1+\cdots+x_k>0$ or not. The asymptotic efficiency (in the sense of Pitman) of the α_k-test with respect to the t-test (assuming normality) is found to be $e_k=(k \arcsin 1/k)^{-1}$. For $k=1$, we have the sign test with asymptotic efficiency $2/\pi$. Further, $e_2=3/\pi$. It turns out that $\alpha_1+\alpha_2$ is Wilcoxon's signed rank test having the same asymptotic efficiency as α_2. {It would appear from the paper that the author is not aware of Wilcoxon's work}. *G. E. Noether* (Boston, Mass.).

Walsh, John E. Asymptotic efficiencies of a nonparametric life test for smaller percentiles of a gamma distribution. J. Amer. Statist. Assoc. 51 (1956), 467–480.

The properties investigated are concerned with the estimation and testing of smaller population percentage points for the case of a sample from a gamma probability distribution. The investigations show that appropriate use of the sign test type of nonparametric tests and estimates sometimes can yield a saving in cost or time without loss of statistical efficiency, since the experiment can be stopped when only a fraction of the items being life tested have failed. The nonparametric results are found to be highly efficient compared to "best" parametric results based on the same fraction of items failed for the case of small population percentage points; this conclusion appears to be valid for any sample size and for any reasonable type of statistical population. (From the author's summary.) *G. E. Noether* (Boston, Mass.).

Savage, I. Richard. Contributions to the theory of rank order statistics — the two-sample case. Ann. Math. Statist. 27 (1956), 590–615.

In the two-sample problem, using ranks, there are certain rank orders that one would expect to be more probable than certain others, purely on an intuitive basis. This paper shows by counterexamples that in general such intuitive orderings need not hold. Sufficient conditions on the alternative hypothesis are given for such orderings to hold. These include Lehmann alternatives for which, for small sample sizes, partial orderings and some power values are explicitly computed. *M. Dwass* (Evanston, Ill.).

Anderson, Oskar. Verteilungsfreie (nicht parametrische) Testverfahren in den Sozialwissenschaften. Allg. Statist. Arch. 40 (1956), 117–127.

Some expository remarks on the role of non-parametric methods in the social sciences. *M. Dwass.*

van Eeden, Constance. A sequential test with three possible decisions for comparing two unknown probabilities, based on groups of observations. Rev. Inst. Internat. Statist. 23 (1955), 20–28.

Let $\{a_i\}$, $\{a_i'\}$ be two sequences of binomial variates with parameters (n_i, p), (n_i', p'), and let $\{u_i\}$, $\{u_i'\}$ be the resulting variates after an arcsin transformation. The sequential procedure of Sobel and Wald [Ann. Math. Statist. 20 (1949), 502–522; MR 11, 261] for the assignment of the mean μ of a normal population into one of the intervals $(-\infty, \mu_0)$, (μ_0, μ_1), (μ_1, ∞), is applied to the variates $x_i=u_i-u_i'$, which are approximately normally distributed with mean

$$\mu=2 \arcsin\{[p(1-p')]^{\frac{1}{2}}-[(1-p)p']^{\frac{1}{2}}\}$$

and known variance

$$\sigma_i^2=(n_i+1)/n_i^2+(n_i'+1)/n_i'^2.$$

I. Olkin (East Lansing, Mich.).

Binet, F. E.; and Watson, G. S. Algebraic theory of the computing routine for tests of significance on the dimensionality of normal multivariate systems. J. Roy. Statist. Soc. Ser. B. 18 (1956), 70–78.

The authors consider the statistical problem of assigning an observed individual to one of several populations, in particular, when there are k p-variate multinormal populations with mean vectors $\mu^{(\pi)}$ $(\pi=1, 2, \cdots, k)$ and a common non-singular covariance matrix Σ. A comparison is made of two methods of solution, the likelihood-ratio approach, and Fisher's generalization of his method of linear discriminant functions. They demonstrate the algebraic identity of the statistics obtained by the two methods. *S. Kullback* (Washington, D.C.).

Soitani, Minoru. On the distributions of the Hotelling's T^2-statistics. Ann. Inst. Statist. Math., Tokyo 8 (1956), 1–14.

The problem considered here is related to the multivariate generalization of the analysis of variance based on

Hotelling's T^2-statistics. An attempt is made to extend the bivariate results of Hotelling [Proc. 2nd Berkeley Symposium Math. Statist. Probability, 1950, Univ. of California Press, 1951, pp. 23–41; MR 13, 479]. Exact distributions are not found for the multivariate case but a formula is provided for the computation of the percentage point of the distribution of T^2, based on an asymptotic series with known terms of order n^{-2}. The corresponding formula for calculating the percentage point of the distribution of the ratio of two T^2-statistics reveals a coefficient of n^{-2} which is very complicated. The author recommends the use of the formula using the n^{-1} term for moderate values of n. *P. S. Dwyer* (Ann Arbor, Mich.).

Grundy, P. M.; Healy, M. J. R.; and Rees, D. H. Economic choice of the amount of experimentation. J. Roy. Statist. Soc. Ser. B. 18 (1956), 32–49; discussion, 49–55.

The problem considered is the choice of the second sample size function, $n_2(\sum_1^{n_1} X_i)$, for double sampling from $N(\theta, \sigma^2)$ with known σ^2, terminal decisions A and with associated regret functions $k'(-\theta)^+$ and $k'\theta^+$, unit sampling cost k, given first sample size n_1 and terminal decision function: A if $\sum_1^{n_1+n_2} X_i > 0$.

The principal result is the explicit construction of the n_2-function Bayes against uniform measure on $(-\infty, \infty)$ and the calculation of representative values of its risk function. Simple transformations of this construction yield the n_2-functions Bayes against normal distributions with zero means. *J. Hannan* (Stanford, Calif.).

Claringbold, P. J. A note on the 4^n series of factorial experiments. Biometrics 12 (1956), 259–263.

The author illustrates the construction of a $\frac{1}{4}$ replicate of a 4^5 fractional factorial design using pseudo factors composed of two levels. Let the two level pseudo factors A', A'' be associated with the four level factors A, etc. Then the identity relationship for the fractional design is

$$I = A'B'C'D'E' = A''B''C''D''E'' = A'A''B'B''C'C''D'D''E'E''.$$

M. Zelen (Washington, D.C.).

Roy, Purnendu Mohon. Analysis of $p\times(p-1)$, n-ple latinized rectangular lattices and their multiples. Calcutta Statist. Assoc. Bull. 6 (1955), 113–131.

A complete analysis of $p\times(p-1,)$ n-ple latinized rectangular lattices and their multiples is given for all n. It is shown that alternative association schemes which are most convenient for the cases with $n > \frac{1}{2}p$ are also available when the complete set of $p\times p$ orthogonal Latin squares exists. *W. S. Connor.*

Gaylor, D. W. Equivalence of two estimates of product variance. J. Amer. Statist. Assoc. 51 (1956), 451–453.

The author proves the equivalence of the covariance technique of Grubbs [same J. 43 (1948), 243–264] and the usual components of variance estimate of the variance of a product apart from measurement variance.
R. L. Anderson (Raleigh, N.C.).

Salvemini, Tommaso. Varianza della differenza media dei campioni ottenuti secondo lo schema di estrazione in blocco. Metron 18 (1956), no. 1–2, 133–161.

A sample from a finite population can be taken either (a) with replacement or (b) without replacement. Gini's mean difference is defined for the sample as well as for the population. The author derives the variance of this sample characteristic if the sampling procedure is (b) and compares it with the (known) variance of the mean difference obtained by method (a). *E. Lukacs.*

Archbold, J. W.; and Johnson, N. L. A method of constructing partially balanced incomplete block designs. Ann. Math. Statist. 27 (1956), 624–632.

The authors describe a new method for constructing partially balanced incomplete block designs. The historical use of finite geometries for this purpose is modified by allowing the coordinate to belong to a linear associative algebra of finite order, over a finite field. Several constructions are displayed. *W. S. Connor.*

Sprott, D. A. A note on combined interblock and intrablock estimation in incomplete block designs. Ann. Math. Statist. 27 (1956), 633–641.

When the block effects in an incomplete block design are random variables, it is possible to use the block totals in the estimation of treatment effects. For certain designs this can be done by applying the method of least squares to the block totals to obtain interblock estimates. Each such interblock estimate may then be averaged with the corresponding intrablock estimate by weighting with the reciprocals of their variances. This is Method 1.

Method 2 consists of a single application of least squares to the following random variables: (i) for each block, the k differences between the observations and the block average, divided by the within blocks variance σ^2; and (ii) the b block totals, divided by the variance of the totals (i.e., by $k^2\sigma_b^2 + k\sigma$, where σ_b^2 is the variance of the block effects). The variance of such estimates is less than or equal to the variance of the corresponding estimates obtained by Method 1. Equality holds only when the two methods yield the same estimate.

The following results are obtained: (a) the two methods yield the same estimates of the treatment effects only for balanced incomplete block designs; (b) for any subset S of treatments in an incomplete block design, the two methods yield the same estimates of differences between effects if and only if all pairs of treatments in S occur in λ_1 blocks and all pairs with exactly one treatment from S occur in λ_2 blocks; (c) if the two methods yield the same estimates for the effects of the treatments in S, then every pair of treatments which contains at least one treatment from S occurs in λ blocks. *W. S. Connor.*

Koyck, L. M. Long-term foreign trade elasticities. A note. Metroecon. 5 (1953), 61–67.

An example of the author's suggested method for estimating distributed lag patterns from time series by assuming that after some point the lag coefficients decrease exponentially. *R. Solow* (Cambridge, Mass.).

Theil, H. The measurement of consumers' behaviour. Rev. Inst. Internat. Statist. 23 (1955), 36–46.

A review article of the recent work of R. Stone [The measurement of consumers' expenditure and behaviour in the United Kingdom, 1920–1938, v. 1, Cambridge, 1954; MR 15, 639]. In commenting on Stone's statistical methods, the author takes occasion to sketch some unpublished developments of his own. Suppose it desired to estimate the parameters γ_μ, β_λ of a structural equation,

$$z(t) = \sum_{\mu=1}^{m} \gamma_\mu y_\mu(t) + \sum_{\lambda=1}^{l} \beta_\lambda x_\lambda(t) + u(t),$$

where z, y_μ are jointly dependent variables in a system of

which the above is one, x_λ are predetermined variables, u a random disturbance, and t time. Let $v(t)$, $v_\mu(t)$ be residuals from the regression of z, y_μ, respectively, on all the predetermined variables in the system. Let $\mathbf{z}$ be the observation vector on the z's, $\mathbf{Y}$ the matrix of observations on the y_μ's, $\mathbf{X}$ the matrix of observations on the x's in the equation, $\mathbf{v}$ the vector of residuals $v(t)$, and $\mathbf{V}$ the matrix of residuals $v_\mu(t)$. For any given scalar k, define vectors $\mathbf{c}$, $\mathbf{b}$ so that,

$$\begin{pmatrix} \mathbf{Y'Y}-k\mathbf{V'V} & \mathbf{Y'X} \\ \mathbf{X'Y} & \mathbf{X'X} \end{pmatrix}\begin{pmatrix} \mathbf{c} \\ \mathbf{b} \end{pmatrix}=\begin{pmatrix} \mathbf{Y'z}-k\mathbf{V'v} \\ \mathbf{X'z} \end{pmatrix}.$$

The vectors $\mathbf{c}$, $\mathbf{b}$ may be taken as estimates of γ, β. For suitably chosen k, the above are the usual simultaneous equations estimates. For $k=0$, the above are least squares estimates. If the disturbances are independent, then the estimates are consistent if k approaches 1 in probability. In general the estimates should have smaller variances (more strictly, a smaller generalized variance) when $k=0$ than for k near 1, but the former are, of course biassed even for large samples. The choice between the two methods is therefore doubtful for small samples.

K. J. Arrow (Stanford, Calif.).

Press, Harry; and Tukey, John W. Power spectral methods of analysis and their application to problems in airplane dynamics. Bell Tel. System Monograph **2606** (1956), i+41 pp.

Notice d'information et de documentation sur les méthodes d'analyse spectrale des phénomènes aléatoires et leurs applications aux problèmes aéronautiques. La première partie est un résumé des principes mathématiques: Séries et intégrales de Fourier; intégrale de Duhamel, notions sur les processus stochastiques, autocovariance et fonction spectrale (spectre de puissance), relation entre les courbes expérimentales et leurs spectres (dans le cas gaussien), étude des systèmes linéaires et de la manière dont ils transforment la fonction spectrale. Le problème de la mesure du spectre de puissance est ensuite abordé, avec examen des divers procédés numériques et discussion des erreurs qui s'introduisent au cours des mesures et des calculs. La seconde partie est consacrée à l'examen plus ou moins détaillé de diverses applications: spectre de la turbulence atmosphérique et son rôle dans le calcul des avions — problème des irrégularités de surface des pistes d'aérodromes — Effets d'amortissement pour des vols en air agité. Effet des variations locales de la turbulence sur la surface d'un avion — Problèmes de "buffeting". Trois appendices sont consacrés aux méthodes numériques, analogiques et approchées de calcul des spectres de puissance.

J. Bass (Paris).

See also: Hall and Swift, p. 192; Teichroew, p. 238; Sarhan and Greenberg, p. 238; Barton and David, p. 240; Hoeffding, p. 240; Geisser, p. 240; Lah, p. 241; Ridderström, p. 242; Klein, p. 265.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

Mişicu, M. Gleichgewicht kontinuierlicher Körper mit endlich grossen Verformungen. Rev. Méc. Appl. **1** (1956), 175–183.

This is a translation into German of part of a paper previously published by the author in Romanian [Acad. R. P. Române. Stud. Cerc. Mec. Metalurgie **4** (1953), 31–53; MR **16**, 307].

J. L. Ericksen (Washington, D.C.).

Istomin, N. V. The tensor of moments of a system of bound vectors and its applications in mechanics. Prikl. Mat. Meh. **20** (1956), 434–438. (Russian)

For any system of n bound vectors

$$\begin{Bmatrix} \mathbf{F}^1 & \mathbf{F}^2 \cdots & \mathbf{F}^n \\ A_1 & A_2 \cdots & A_n \end{Bmatrix}$$

the author defines the moment with respect to a plane as the sum of n products of the given vectors $\mathbf{F}^\nu$ by the distances of their points of application from the plane. This definition is a generalisation of Monge's definition of the moment of a system of parallel bound vectors with respect to a plane [cf. Appell, Traité de mécanique rationnelle, t. I, 5ième éd., Gauthier-Villars, Paris, 1926, p. 43]. For the coordinate planes of a Cartesian coordinate system $Ox^1x^2x^3$ the so defined moments are

$$\mathbf{S}_1=x_\nu{}^1\mathbf{F}^\nu,\quad \mathbf{S}_2=x_\nu{}^2\mathbf{F}^\nu,\quad \mathbf{S}_3=x_\nu{}^3\mathbf{F}^\nu.$$

Projecting these vector equations on the coordinate axes of the given coordinate system we obtain

$$S_{ij}=x_\nu{}^iX_j{}^\nu \quad (i,j=1,2,3;\ \nu=1,2,\cdots,n),$$

where $\mathbf{F}^\nu=\{X_1{}^\nu, X_2{}^\nu, X_3{}^\nu\}$. The Cartesian tensor $\{S_{ij}\}$ is then defined as the tensor of moments of a system of bound vectors.

The author shows that this notion of the tensor of moments can be successfully applied to the investigation of properties of the so called Hamilton's centre of any system of bound vectors. The following concrete questions are discussed: 1) the behaviour of Hamilton's centre in the case of rotation of all vectors of the system about the parallel axes through the respective points of application, and 2) the existence of such systems of bound vectors whose Hamilton's centre have the properties of the corresponding centre of parallel vectors or the properties of the centre of a plane vector system.

T. P. Anđelić (Belgrade).

Balescu, R. Le théorème de Poincaré pour un ensemble d'oscillateurs. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 622–627.

The starting-point in this paper is a theorem of Poincaré [Les méthodes nouvelles de la mécanique céleste, t. I, Gauthier-Villars, Paris, pp. 233–252, 1892] about the existence of the uniform and analytic integrals for the following canonical normal systems of differential equations

$$\frac{dx_i}{dt}=\frac{\partial F}{\partial y_i},\quad \frac{dy_i}{dt}=-\frac{\partial F}{\partial x_i},$$

where F is a function of x_i, y_i and of a parameter μ, which for small values of μ can be written in the form

$$F=F_0+\mu F_1+\mu^2F_2+\cdots$$

with F_0 independent of y_i. The author gives the conditions under which a system of slightly anharmonic oscillators, i.e. of a slightly perturbated system of harmonic oscillators, does not admit uniform and analytic integrals of motion which differ from the Hamiltonian (i.e. from the energy integral) in the sense of the above

mentioned theorem of Poincaré. He then applies the obtained results to Peierls' system of oscillators on a lattice [Ann. Physik (5) 3 (1929), 1055–1101].

{The author emphasizes that the original proof of Poincaré is not applicable to the case of slightly anharmonic oscillators, but we must observe that we find no essential difference between his and Poincaré's proof except perhaps in the way of deriving the conclusion. It follows directly from a comparison of the Poincaré's equations with the corresponding equations of the author.}

T. P. Anđelić (Belgrade).

Sparenberg, J. A. Oscillator with an amplitude bounded at one side. Appl. Sci. Res. A. **6** (1956), 53–66.

The system considered is an oscillator, consisting of a mass and a linear spring, with the novel feature that a fixed wall prevents the excursion of the mass in one direction from exceeding a certain value $\bar{x}_0$. It is assumed that the collisions of the mass with the wall are instantaneous and perfectly elastic. As far as the free motions are concerned, one effect of the presence of the wall is to diminish the period of the periodic motions with large amplitude. The dependence of period upon amplitude is studied. Most of the paper is concerned with periodic motions in response to a sinusoidal applied force, and in particular with motions having the same period as has the force. It is shown that there cannot be more than two such motions, and certain calculations indicate that there is probably only one. Experiments are described which confirm many of the theoretical results.

L. A. MacColl (New York, N.Y.).

Lawden, D. F. Optimal programming of rocket thrust direction. Astronaut. Acta **1** (1955), fasc. 1, 41–56.

The author explores the problem of optimal utilization of fuel over a given time interval, for the general case of a rocket moving in a gravitational field of specified potential, in terms of instantaneous direction of thrust. He concludes by examination of general principles and numerically integrated tables, that in the case of a rocket moving in vacuo in a uniform gravitational field, it is advantageous to displace the line of thrust from the direction of motion by a small angle which diminishes to zero as the maneuvre proceeds. The best value for the initial angle of throw-off for optimal escape from a circular orbit is left for later study. *A. A. Bennett.*

See also: Krause, p. 261.

Statistical Mechanics

Nakajima, S. Perturbation theory in statistical mechanics. Advances in Physics **4** (1955), 363–380.

This is primarily a review article. Two derivations are given for the pertubation expansion in powers of the free energy ε for a quantum mechanical system having a Hamiltonian of the form $H=H_0+\varepsilon H_1$. The use of these formulae is illustrated by applying them to the electron-phonon interaction in non-superconducting metals and the interaction with the magnetic field of a charged gas (diamagnetism). The author succeeds in bringing several topics together in a unified and concise way.

G. Newell (Providence, R.I.).

Kaeppeler, H. J. Über den Transport von Translationsenergie bei Kurzzeitvorgängen. Astronaut. Acta **1** (1955), fasc. 3, 111–119.

A problem of the following type is formulated and partially solved. Some localized molecules of a gas suddenly acquire an excess of translational energy as a result of a chemical reaction, for example. In the process of conducting this energy by molecular collision, some of the translational energy is lost due to the excitation of relatively stable rotational and vibrational states of the molecules. The author finds, as a consequence of this, that there is a decrease in the effective coefficient of conduction for translational heat energy. *G. Newell.*

See also: Bopp. p. 259.

Elasticity, Visco-elasticity, Plasticity

Bašeleišvili, M. O. Solution of the first fundamental boundary problem of statics for an orthotropic elastic body in the case of multiply connected domains. Soobšč. Akad. Nauk Gruzin. SSR **16** (1955), 577–582. (Russian)

The author formally solves the first fundamental problem for plane regions by reduction to a Fredholm equation for the case of finite as well as infinite regions.

J. R. M. Radok (Emeryville, Calif.).

Manacorda, Tristano. Sul potenziale isotermo nella più generale elasticità di secondo grado per solidi incomprimibili. Ann. Mat. Pura Appl. (4) **41** (1956), 77–86.

Considering an elastic potential having one more arbitrary constant than does Mooney's form, the author finds necessary and sufficient conditions that this potential be positive definite. He also finds restrictions necessary and sufficient that the signs of the derivatives of the tensile stress in simple tension be those experimentally observed. *C. Truesdell* (Bologna).

Signorini, Antonio. Trasformazioni termoelastiche finite. III. Solidi incomprimibili. Ann. Mat. Pura Appl. (4) **39** (1955), 147–201.

Continuing the author's general exposition of the theory of finite elastic strain [same Ann. (4) **22** (1943), 33–143; **30** (1949), 1–72; MR **8**, 240, 708; **11**, 756], this paper concerns incompressible bodies. Most of the considerations run parallel to the author's earlier work on compressible bodies or present in the author's notations results already available in the international literature. *C. Truesdell.*

Wilde, Piotr. Curvilinear girders of thin-walled open cross-section elements. Arch. Mech. Stos. **8** (1956), 41–50. (Polish. Russian and English summaries)

Wilde derives the differential equations for the deflection curve and angle of twist of a curvilinear girder, preserving the terms which are usually neglected in the equation for the deflection curve of a straight beam. This is applied to a numerical example of a circular arc girder simply supported and continuously loaded with the corresponding boundary conditions. Two ways are used: Calculating the constants of the general solution from the boundary conditions and the use of a trigonometric series. The author shows that in some cases for large angles of the arc (90^0) the error in the magnitude of the bimoment in the middle of the span of the girder when

using the equations for the straight beam reaches the value of 22 percent, hence is not negligible.

M. Z. Krzywoblocki (Urbana, Ill.).

Kondrat'ev, A. S. Asymptotic formulas for forms of equilibrium of a bar under longitudinal bending. Kuĭbyšev. Indust. Inst. Sb. Nauč. Trudy. **1953**, no. 4, 3–12. (Russian)

The problem considered is a special case of the Sturm-Liouville problem of deriving asymptotic formulas for eigenfunctions for the differential equations of longitudinal bending in the case of variable rigidity. It is proved that for the equation

$$u''+\lambda u=f(t)u+f_1(\lambda)\varphi(t)+f_2(\lambda)$$

the solutions are bounded for increasing parameter. In this it is assumed that the ratios $\lambda^{-\frac{1}{2}}f_1(\lambda)$, $\lambda^{-\frac{1}{2}}f_2(\lambda)$ are bounded and also that the boundary condition and the normalizing condition $u(0)=u'(0)$ and $\int_0^l u^2dt=1$ hold. The consideration of the above differential equation made it possible for the author to obtain asymptotic formulas for the equilibrium shape in the case of clamped ends and those equilibrium shapes in which transverse reaction appear in the framing.

It is indicated that the formulas obtained for the equilibrium shapes are applicable to finding the abscissas of nodes and antinodes for sufficiently large numbers.

L. N. Ter-Mkrtič'yan (RŽ Meh 1955 No. 3474).

Kapanyan, L. K. On the bending of certain hollow console rods. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **9** (1956), no. 3, 33–43. (Russian. Armenian summary)

Consider the bending of a prismatic console rod by a force attached at the end of it. The force acts in the plane passing through one of the principal axes of inertia of the cross section. The problem of finding the stress given distribution reduces to that of determining the stress function by a linear nonhomogeneous partial differential equation of the second order. After some manipulations the author assumes this function in terms of a multiple complex variable function, in particular in the form of a logarithmic function of a complex variable. To define a constant which appears in the expression for the stress function, the author applies the circulation theorem which states that in case of pure bending the circulation is equal to zero. Having the stress function the stresses can be determined from it by means of elementary differentiation. To determine the stress function, the author first applies the conformal mapping. This maps both boundary curves of the cross section of the hollow rod onto two concentric circles. The mapping function is represented by means of an infinite power series with unknown coefficients. These are determined from the boundary conditions in the transformed plane of two circles. The problem reduces to a system of algebraic equations. The general theory is applied to two particular cases: (i) external boundary curve of the cross section is a circle; the internal is a square; the force acts in the direction of the diagonal of the square; (ii) the same, the force acts in the middle of a side of the square. The numerical calculations are given up to the sixth decimal place.

M. Z. Krzywoblocki (Urbana, Ill.).

Berger, E. R. Die Variationsprinzipien der Elastostatik in der Theorie zweiter Ordnung. Österreich. Ing.-Arch. **10** (1956), 124–129.

This paper deals with finding a complementary (Castigliano) principle in certain elastic problems where the strain energy is not a quadratic functional of the displacements. For the sake of definiteness, the one-dimensional problem of a beam subject to pressure and bending is considered. The usual Hooke's law strain energy in terms of longitudinal strain and curvature is taken, but the strain is approximated by a quadratic rather than a linear form in the displacement gradients.

The method of K. O. Friedrichs [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1929**, 13–20; Courant and Hilbert Methoden der mathematischen Physik, Bd. 1, 2nd ed., Springer, Berlin, 1931, pp. 201–209] is applied to the minimum energy principle to yield the desired complementary energy principle.

H. Weinberger.

Gorgidze, A. Ya. On secondary effects in the problem of bending of a prismatic beam by a transverse force. Soobšč. Akad. Nauk Gruzin. SSR **16** (1955), 665–672. (Russian)

The author deduces equations for the problem of bending of compound bars by transverse forces for a non-linear stress strain law.

J. R. M. Radok.

Negoro, Shosaburo. On the elastic failure and the buckling of a column under eccentric loads. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **4** (1955), 41–56.

Buckling of hinged bars of uniform cross-section is discussed to show that the effects of eccentric loading can be considerable. The two cases of the principal axes within or outside the plane of loading are treated by assuming that failure occurs when the maximum skin stress becomes equal to the yield stress. A number of approximations are made and the results are compared with those given by Euler's buckling.

B. R. Seth.

Nowacki, Witold. Assemblage stresses in plates. Arch. Mech. Stos. **8** (1956), 215–232. (Polish. Russian and English summaries)

The assemblage stresses in a plate appear when the supports lie in the same plane but the plate is not perfectly flat. The stresses caused by that can be calculated from an integral functional derived from the principle of virtual work. Using Green's theorem this can be represented in terms of line integrals. The author derives the final formulas for the deflections of simply supported or rigidly fixed plates, which formulas are very simple. A plate simply supported on part of its circumference and rigidly fixed on the remaining part furnishes a more complicated system of integral equations. The theory is applied to numerical examples of a rectangular plate freely supported and rigidly supported having given initial deflections and to a semi-infinite strip.

M. Z. Krzywoblocki (Urbana, Ill.).

Kalynyak, N. I. Bending of a thin isotropic plate with two equal circular openings. L'vov. Politehn. Inst. Nauč. Zap. **30**, Ser. Fiz.-Mat. No. 1 (1955), 52–64. (Russian)

The author uses bipolar coordinates and solves the transformed differential equation by use of infinite series, the coefficients of which can be determined successively.

J. R. M. Radok (Emeryville, Calif.).

Karas, K. Die Schirmschwingungen der Kreismembran unter allgemeinen Bedingungen. Österreich. Ing.-Arch. **10** (1956), 200–220.

Elastic properties in the plane of the "membrane" are

considered, and the membrane has a loaded edge. Its equation of motion is derived and boundary conditions are given. The problem is put in integro-differential equation form. This is specialized in various ways. The Ritz method of numerical solution is given.

E. Pinney (Berkeley, Calif.).

Higashi, Yôichi. Bending of thin rectangular plates with any boundary conditions. Mem. Fac. Tech. Tokyo Metro. Univ. no. 6 (1956), 359–391.

The present paper gives six types of Fourier series solutions for the deflection of a thin rectangular plate with any transverse load and with any boundary conditions in the most general form. Anyone of the types can be chosen conveniently according to the given boundary conditions. The slopes, the bending moments, the shear forces and simliar quantities can be calculated by taking appropriate derivatives of the solutions, which converge uniformly and absolutely over the region bounded by the plate including the edges and the corners except at points subjected to the line loads or to concentrated loads and at points of discontinuity of the boundary condition. An infinite set of equations with an infinite number of unknown coefficients obtained using the solution given in this paper must however be solved to obtain a solution in the case of more general boundary conditions.

R. Gran Olsson (Trondheim).

Nowacki, Witold; and Olesiak, Zbigniew. The problem of a circular plate partially clamped and partially simply supported along the circumference. Arch. Mech. Stos. 8 (1956), 233–255. (Polish. Russian and English summaries)

The authors consider a general problem of forced vibrations of a circular plate subject to a periodic load normal to the plane of the plate and a steady compression load, g, in the middle plane of the plate. For mixed boundary conditions (circumference partly simply supported partly fixed) the authors assume the superposition of deflections originated from uniform boundary conditions (totally simply supported or totally fixed). The total deflection is sought in the form of an integral. This transforms the original partial differential equation for the total deflection into an integral Fredholm equation of the first kind. The kernel of this equation is found in terms of Bessel functions and the Fredholm equation is solved by representing the integral in form of a sum. This gives a finite system of linear nonhomogeneous algebraic equations. Particular cases treated are: forced vibrations ($g=0$), fundamental frequency of vibrations ($g=0$), combined bending and compression, buckling and bending. The problem can be generalized to a plate clamped along r arc segments on the circumference and simply supported along the remaining arcs. The authors present also a rigorous solution of the Fredholm equation leading to an infinite system of algebraic equations.

M. Z. Krzywoblocki (Urbana, Ill.).

Alumyaè, N. A. On the representation of the fundamental relations of the nonlinear theory of shells. Prikl. Mat. Meh. 20 (1956), 136–139. (Russian)

The author deduces the basic relations of the nonlinear theory of shells in a form equivalent to that introduced into the linear theory by V. V. Novožilov [Theory of thin shells, Gos. Izdat. Sudostroitel'noĭ Lit., Moscow, 1951; MR 17, 915]. The underlying principle is to stress the formal analogy between the equilibrium and compatibility equations involving the stress and strain tensors respectively.

J. R. M. Radok.

Ševlyakov, Yu. A. On the concentration of stress about an opening in a cylindrical shell. Dnepropetrov. Gos. Univ. Nauč. Zap. 41 (1953), 79–91. (Russian)

The author considers the problem of concentration of stresses in a cylindrical shell about openings; the discussion is based on the equations of V. Z. Vlasov [General theory of shells and its applications in technology, Gostehizdat, Moscow-Leningrad, 1949; MR 11, 627].

The function of stresses F and the bending are taken in the form

$$F=F_0+\xi F_1+\cdots,\quad w=w_0+\xi w_1+\cdots,$$

where $q=h/R$ is a small parameter, h is the thickness, and R is the radius of the middle surface. The following equations are obtained:

$$\nabla^4F_0=0,\ \nabla^4F_1=-E\frac{\partial^2w_0}{\partial x^2},\ \nabla^4F_2=-E\frac{\partial^2w_1}{\partial x^2},\ \cdots,$$

$$\nabla^4=w_00,\ \nabla^4w_1=\frac{1}{Dh}\frac{\partial^2F_0}{\partial x^2},\nabla^4w_2=\frac{1}{Dh}\frac{\partial^2F_1}{\partial x^2},\ \cdots.$$

Here

$$\nabla^4=\frac{\partial^4}{\partial x^4}+\frac{2}{R^2}\frac{\partial^4}{\partial x^2\partial\varphi^2}+\frac{1}{R^4}\frac{\partial^4}{\partial\varphi^4},\ D=\frac{Eh^3}{n(1-\mu^2)}.$$

In this way the problem of equilibrium of a cylindrical shell reduces to a sequences of plane problens and a sequence of problems of bending of thin plates. The author uses for the functions $F_0, F_1, F_2, \cdots, w_0, w_1, w_2, \cdots$ known expressions in terms of functions of a complete variable. These functions satisfy boundary conditions on the contour of the opening that are identical to those that occur in the plane problem and in the problem of bending of thin plates. No examples are given to illustrate the method.

{However, the reduction of the problem of equilibrium of a cylindrical shell to the problem of equilibrium of a plane plate using "the method of small parameter" inevitably leads to the result that the functions F_1, F_2, $\cdots$, w_1, w_2, $\cdots$ and the corresponding forces and moments will grow beyond all bounds as we move away from the opening, and it does not appear possible to get rid of the singularities at infinity. One can convince oneself of this by considering even the simplest case, namely the stretching of a cylindrical shell with a circular opening; it is easy to do this on the basis of the formulas and equations derived in the paper. In view of this the question of the availability of the proposed "method of small parameter" for the solution of problems of this kind remains open.}

E. F. Burmistrov (RŽ Meh 1953–54 No. 2240).

Nowacki, W.; and Olesiak, Z. The stability of a cylindrical shell with ribs. Rozprawy Inż. 4 (1956), 3–22. (Polish. Russian and English summaries)

The results of the theory of shells in bending and compression developed by V. Z. Vlasov [General theory of shells and its applications in technology, Gostehizdat, Moscow-Leningrad, 1949; MR 11, 627] are applied by the authors to a case of a cylindrical shell with longitudinal and transverse ribs. The solution of the differential equation is achieved by means of a double trigonometric series. The condition of the equal deflections of the ribs and of the skin leads to the system of equations whose determinant must be

equal to zero. This gives the buckling conditions. The case of the four freely supported edges is easier and can be solved in a rigorous way. The case of one edge built in and three others freely supported is more complicated. The general theory is applied to various particular cases: a shell with transverse ribs only, (in some simple cases one or two ribs), a shell with one longitudinal rib; uniformly distributed load and forces acting only on the ribs. The latter case is represented by means of eight diagrams. The case of a shell with a dense net of ribs is treated by means of an orthotropic model. No numerical examples are given.
M. Z. Krzywoblocki (Urbana, Ill.).

Olszak, W.; and Urbanowski, W. Thick-walled elasto-plastic spherical shell of non-homogeneous material, subjected to internal and external pressure. Rozprawy Inż. 4 (1956), 23–41. (Polish. Russian and English summaries)

The authors consider a thick-wall spherical shell (radii $a<b$) subject to internal and external pressures, p and q, under the assumption that the difference $(p-q)$ is an increasing function of time. They introduce two functions E_0 (mean elongation) and Z_0 (function of the radial displacement and radial coordinate) and derive expressions for the strain and stress deviators in terms of E_0, Z_0, modulus G and Poisson ratio ν. Assumption is made that the non-homogeneity is of the type for which one can take $\nu=$const. The equilibrium conditions furnish a system of two equations for the unknown functions E_0 and Z_0. The stresses in the elastic region (external) are determined from the assumption that in this state the material is incompressible ($E_0=0$) which furnishes the value $\nu=\frac{1}{2}$. Introducing the boundary conditions in this region, one obtains a system of equations for the stress components in the elastic region. In the plastic region the authors introduce a yield condition for the plastically inhomogeneous material. With $E_0=0$ and $\nu=0.5$, they derive a system of equations for the stress components in the plastic region. The value of the pressure between both regions is determined from the expression for a smooth transition from one region into another. The radius of the sphere of the boundary between these two regions is obtained from the condition that the mean normal stress component should be a smooth curve when crossing the boundary between two regions. The formulas derived above allow one to find the values of the pressure difference $(p-q)$ corresponding to the first plastic deformation on the internal radius of the sphere (first critical state) and that corresponding to the plastic deformation of the whole sphere (second critical state). In the last part the authors discuss the conditions for occurrence of the plastic state simultaneously in the whole shell.
M. Z. Krzywoblocki (Urbana, Ill.).

Rutecki, Jerzy. The torsion of a thin-walled rectangular tube beyond the elastic limit. Arch. Mech. Stos. 8 (1956), 29–40. (Polish. Russian and English summaries)

The author considers the problem of torsion of a thin-walled rectangular tube beyond the elastic limit assuming the geometry of the deformation of the cross-section and the condition that the plastic state is reached at all points of the cross-section. The normal and shearing stresses satisfy the condition of equilibrium and the yield hypothesis of Huber-Mises-Hencky and are in agreement with the stress diagrams for the elastic state. This enables the author to derive relatively simple formulas for the resulting stresses and cross-section area of the tube depending upon the bimoment, the bending-twisting moment and the twisting moment. Since the assumption on the plastic state at all points of the cross-section does not occur in reality, the author estimates the error in the stress distribution equal to 3.8 percent.
M. Z. Krzywoblocki (Urbana, Ill.).

Nowiński, Jerzy. Application of the Laplace transformation to problems of torsion of thin-walled tubes. Arch. Mech. Stos. 8 (1956), 111–119. (Polish. Russian and English summaries)

Applying the Laplace transform to the results of W. Z. Vlasov's "Theory of thin-walled tubes" [Stroiizdat, Moscow, 1940], the author derives the equation for the angle of twist of a thin-walled tube subject to the action of a torque. Two cases are considered: uniformly distributed torque, m_0, and a concentrated torque. The numerical example in the first case gives the magnitude of the angle of twist in the middle of a tube supported at both ends equal to 0.209 m_0 (Ritz's method gives 0.211 m_0). *M. Z. Krzywoblocki* (Urbana, Ill.).

Masur, E. F. An extended upper bound theorem on the ultimate loads of buckled redundant trusses. Quart. Appl. Math. 14 (1956), 315–317.

In an earlier paper [same Quart. 11 (1954), 385–392; MR 15, 759] the author made possible the determination of sets of upper and lower bounds on the ultimate loads of buckled redundant trusses, such that the true ultimate load is in the set of calculable lower bounds but is not in the set of upper bounds. The present note defines an extended calculable set of upper bounds which includes the ultimate load. *F. B. Hildebrand* (Cambridge, Mass.).

Rozovskiĭ, M. I. Radial vibrations of a hollow sphere with the singular kernel of elastic after-effect. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 672–675. (Russian)

Elastic after-effects are taken into account by replacing Lamé's constants λ, μ by $\bar{\lambda}=\lambda(1+\phi^*)$, $\bar{\mu}=\mu(1+\psi^*)$, where ϕ^* is a functional of Volterra's type

$$\phi^* y=\int_0^t \phi(t,\tau)y(\tau)d\tau$$

and ψ^* is similar. An attempt is made to solve the integro-differential equation

$$(\bar{\lambda}+2\bar{\mu})\ \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}+2\frac{u}{r}\right)=\rho\frac{\partial^2 u}{\partial t^2}$$

in the form

$$u(r,t)=\sum T_k(t)\Phi_k(r).$$

The formal work leading to a series for $T_k(t)$ seems to be incorrect. *R. C. T. Smith* (Armidale).

Kączkowski, Zbigniew. Free vibrations and buckling of a triangular plate. Arch. Mech. Stos. 8 (1956), 13–28. (Polish. Russian and English summaries)

The author considers some cases of free vibrations and buckling of rectangular and triangular plates, resting on an elastic foundation, with simply supported or clamped edges and loaded in their planes with uniformly distributed normal forces. He uses the technique of expressing the deflection by means of a double trigonometric series jointly with calculating the roots of the determinant to

obtain the first (natural) frequency, second, etc. The paper is limited to giving general forms of equations with no numerical example. *M. Z. Krzywoblocki.*

Pólya, Georges. Sur quelques membranes vibrantes de forme particulière. C. R. Acad. Sci. Paris **243** (1956), 469–471.

Let λ_k and μ_k be the kth eigenvalues of the fixed and free membrane problem, respectively, on a bounded plane domain D of area A. In a previous note [same C. R. **242** (1956), 708–709; MR **17**, 628] the author has shown that if the whole plane can be paved by domains congruent to D, then

$$(1) \qquad \lambda_k \geqq 4\pi k/A \geqq \mu_{k+1}.$$

The bound $4\pi k/A$ is Weyl's asymptotic form for both λ_k and μ_k.

The author defines D to be a quasi-trapezoid if part of its boundary consists of two segments of parallel lines at distance L from each other, and if an infinite strip of width h can be paved by domains congruent to D. For such domains, the following bounds are proved:

$$(2) \qquad \lambda_k \geqq \pi^2 L(kh^2/A)/h^2 > 4\pi k/A > \pi^2 M(kh^2/A) \geqq \mu_{k+1},$$

where the functions $L(s)$ and $M(s)$ are defined by

$$(3) \qquad s = \sum_{1 \leq n \leq L} \sqrt{(L-r^2)}, \quad s = \sum_{0 \leq n \leq M} \sqrt{(M-r^2)}.$$

H. Weinberger (College Park, Md.).

Kleeman, P. W. The buckling shear stress of simply-supported, infinitely-long plates with transverse stiffeners. Aero. Res. Council, Rep. and Memo. no. **2971** (1953), 18 pp. (1956).

This paper is an extension of previous theoretical investigations of the elastic buckling in shear of flat plates reinforced by transverse stiffeners. The plates are treated as infinitely long and simply supported along the long sides. Stiffeners are spaced at regular intervals, dividing the plate into a number of panels of uniform size. The effect of bending and torsional stiffnesses of the stiffeners upon the buckling shear stress is calculated for the complete range of stiffnesses, for panels with ratios of width to stiffener spacing of one, two and five. The results are presented in tabular and graphical forms.

R. Gran Olsson (Trondheim).

Olszak, W.; and Urbanowski, W. The orthotropy and the non-homogeneity in the theory of plasticity. Arch. Mech. Stos. **8** (1956), 85–110. (Polish. Russian and English summaries)

The authors consider a special case of anisotropy i.e., orthotropy, referring to three mutually perpendicular planes of symmetry. In particular they take into account non-homogeneous bodies of curvilinear orthotropy. In this case the expression for the flow function simplifies considerably. The general theory is applied to two-dimensional problems of plastic yielding in the case of plane state of strain and that of plane state of stress. Using the notion of the stress function the problem is reduced to a non-linear partial differential equation, which is derived in various coordinate systems: cartesian, polar, bipolar, elliptic and parabolic. As a particular case they investigate a thick-wall cylinder of a polar orthotropy and axially symmetrical non-homogeneity. The appendix contains the expression for the stress function in plane Lamé coordinates *M. Z. Krzywoblocki* (Urbana, Ill.).

Bishop, J. F. W. An approximate method for determining the temperatures reached in steady motion problems of plane plastic strain. Quart. J. Mech. Appl. Math. **9** (1956), 236–246.

Paper is concerned with approximate numerical solution of heat conductivity equation in two space dimensions and time. Author assumes that effects of convection and heat generation, calculated in advance from known velocity and stress fields, can be taken over a short time interval occurring spontaneously. He then solves the heat conduction equation over this time interval. The procedure is repeated. An extrusion example indicates accuracy of method is of order 10%.

D. R. Bland (London).

Piątek, Marian. Dynamical stability of axially loaded bars with arbitrarily variable cross-section. Arch. Mech. Stos. **8** (1956), 51–68. (Polish. Russian and English summaries)

The author considers the problem of dynamical stability of a pin-jointed bar loaded by an axial pulsating force. The mass and the bending stiffness of the bar are variable along its length and are represented by means of trigometric series. Applying the condition of orthogonality the author reduces the differential equation for the deflection curve to the Hamilton variational principle, whose integrals are representable in form of Stieltjes integrals (called Radon's integrals). This results in a functional whose Euler-Lagrange differential equation (called Hamilton-Ostrogradski formula) solves theoretically the problem. In practical applications some approximation procedure must be used. In the step procedure used by the author the Euler-Lagrange equation reduces to a system of Hill's differential equations, from which one obtains Euler's formula for the critical load. Similarly, the equations for the frequency reduce to the Mathieu differential equations. The stability of the solution is verified by means of Strutt charts. The coefficients in trigonometric expansions are determined by a numerical method. An example of a long frustum under the axial load shows that the difference between the second and third approximation is negligible.

M. Z. Krzywoblocki (Urbana, Ill.).

Freiberger, Walter; and Tekinalp, Bekir. Minimum weight design of circular plates. J. Mech. Phys. Solids **4** (1956), 294–299.

The paper is concerned with the minimum weight design of simply supported circular plates under rotationnally symmetric loading. The plate material is rigid-plastic and obeys v. Mises's yield condition and flow rule. The authors set up a variational problem with two subsidiary conditions. Euler equations are then used to show that the minimum weight design admits a failure mechanism for which the mechanical energy dissipated per unit volume has a constant value throughout the plates. The distribution of the bending moments and thickness throughout the plates are also determined and the results are compared with previous work on the minimum design of plates composed of material obeying Tresca's yield condition [Hopkins and Prager, J. Appl. Mech. **22** (1955), 372–374]. *E. T. Onat* (Ankara).

Hodge, P. G., Jr.; and Romano, F. Deformations of an elastic-plastic cylindrical shell with linear strain hardening. J. Mech. Phys. Solids **4** (1956), 145–161.

The paper is concerned with the deformations of a

cylindrical shell subjected to a slowly increasing radial loading. The shell is composed of an elastic-linear strain hardening material. The absence of end load enables the authors to express the problem in terms of two stress resultants and the corresponding circumferential extension and axial curvature of the middle surface. The concepts of the theory of plasticity are then applied to these generalized components of stress and strain. The initial yield curve of Drucker [Proc. 1st Midwestern Conference Solid Mech., Univ. of Illinois, Urbana, Ill. 1953, pp. 158–163] is approximated by a square and it is assumed that the current yield curve is again a square which differs only in size. For the problems discussed in the paper it is assumed and verified a posteriori that only one face of the yield square is operative. Under these assumptions the mathematical problem reduces to the solution of a simple fourth-order linear equation with constant coefficients. The theory is applied to an example. Various limiting cases are considered including elastic perfectly plastic and rigid strain hardening. *E. T. Onat* (Ankara).

Vargo, Louis G. Nonlinear minimum-weight design of planar structures. J. Aero. Sci. 23 (1956), 956–960.

Methods of limit analysis have repeatedly been applied to the minimum-weight design of beams or plane frames under the assumption that the weight W per unit length of a structural member is proportional to its fully plastic bending moment M. Actually, W is more nearly proportional to M^a with $a<1$. For instance, if the considered sections are geometrically similar, strict proportionality obtains with $a=\frac{2}{3}$. The paper is concerned with the consequences of the adoption of this more realistic relation between W and M. {A similar discussion was given by the reviewer in J. Engrg. Mech. Div. Proc. Amer. Soc. Civil Engrs. 82 (1956), no. EM4, paper 1073.} *W. Prager.*

Olszak, Wacław; Murzewski, Janusz; and Golecki, Józef. Non-homogeneous elastic-plastic semi-infinite plate loaded by a concentrated force. Arch. Mech. Stos. 8 (1956), 197–214.

The paper is concerned with a problem in plane strain: the half-plane subjected to a single load acting normal to the (horizontal) boundary. The elastic, plastic material of the inhomogeneous half-plane is assumed to obey a stress-strain law of the finite type that corresponds to linear work-hardening. The elastic moduli, the yield limit, and the workhardening coefficients, are assumed to be proportional to the same function $g(x)$ of the depth x. Using polar coordinates with the origin at the point of application of the load, the authors establish the forms of $g(x)$ for which a purely radial stress system is possible. For an incompressible material $g(x)=x$ and $g(x)=x/(x+\text{const})$ are found to be the only possibilities. The solution obtained for the incompressible material is adapted to a material that exhibits compressibility in the elastic but not in the plastic range. An approximate solution is given for a material that is compressible in both the elastic and the plastic range. *W. Prager.*

Prager, William. Thermal stresses in viscoelastic structures. Z. Angew. Math. Phys. 7 (1956), 230–238.

Thermo-mechanical behaviour of statically indeterminate trusses consisting of Maxwell bars is discussed under typical variations of loads and temperatures. By introducing two orthonormal states S' and S'' defined by

$$\sum cS'S''=0,\ \sum cS'^2=1,\ \sum cS''^2=1,$$

c being the elastic compliance, an m-parametric system is treated as a linear combination of m orthonormal loads.

For homogeneous stress in which the relaxation time has the same value for each bar two particular cases of intensity-coefficients are discussed — one obeys the classical elastic-plastic linear relation and the other the sinusoidal.

For inhomogeneous trusses the characteristic states of residual stress are used. It is found that the thermal effects produced resemble those in a homogeneous truss except that the various states of residual stress have to be associated with different relaxation times. The particular example of combined load and temperature effects varying as cos wt and sin wt give interesting results. *B. R. Seth* (West Bengal).

See also: Tyutekin, p. 256; Heaps, p. 263.

Fluid Mechanics, Acoustics

Bhattacharya, S. K. Streamlined flow past an obstacle bounded by two intersecting circles. J. Assoc. Appl. Phys. Calcutta Univ. 3 (1956), 5–13.

The author uses coaxial coordinates defined by $z=ia\coth\zeta/2$, $\zeta=\xi+i\eta$ to show that the complex potential

$$w=(2aV/n)[\coth(\zeta/n)\sin\alpha+i\ \mathrm{cosech}\ (\zeta/n)\cos\alpha]$$

gives the flow when the stream $-Ve^{i\alpha}$ is disturbed by the cylinder whose cross-section is the configuration of two equal intersecting circles for which $\eta=\pm n\pi/2$. Detailed applications are made to the cases $n=5/3$, $n=1/2$ which correspond with the diagrams of Fig. 6.51(ii) in the reviewer's "Theoretical hydrodynamics" [3rd ed., Macmillan, New York, 1956; MR 17, 796]. *L. M. Milne-Thomson* (Providence, R.I.).

Mackie, A. G. An application of Hankel transforms in axially symmetric potential flow. Proc. Edinburgh Math. Soc. (2) 9 (1956), 128–131.

The paper deals with flow through a circular aperture in an infinite plane wall, the motion being irrotational and the liquid inviscid. If z is the axis and r denotes distance from the axis, it is shown that

$$\varphi=\int_0^\infty \xi F(\xi)e^{\pm\xi z}J_0(\xi r)d\xi$$

satisfies Laplace's equation. The author succeeds in finding two potential functions φ_1 for $z\leqq 0$ and φ_2 for $z\geqq 0$, such that φ_2 continues φ_1 through the aperture and on the side $z>0$ the liquid forms a jet tending to constant speed U and constant radius at infinity. For given U these flows form a one parameter family. *L. M. Milne-Thomson* (Providence, R.I.).

Donoughe, Patrick L. Analysis of laminar incompressible flow in semiporous channels. NACA Tech. Note no. 3759 (1956), 25 pp.

Perturbation solutions of equations for laminar incompressible flow in a semiporous channel are presented, and the results are compared with those obtained from a fully porous channel. (From the author's summary.) *R. Finn* (Pasadena, Calif.).

Schmieden, Curt; und Müller, Karl-Heinz. Die Strömung einer Quellstrecke im Halbraum, eine strenge Lösung der Navier-Stokes-Gleichungen. Z. Flugwiss. **4** (1956), 300–309.

The steady flow of an incompressible viscous fluid due to a line-source of constant strength lying perpendicular to a plane stationary wall is postulated. Far from the wall the flow must approach plane irrotational source flow. The Navier-Stokes equations, written in terms of Stokes' stream function, are reduced to an ordinary, non-linear, fourth-order equation which can be integrated to yield Ricatti's equation. This is linearized by a simple change of dependent variable, and has solutions involving hypergeometric functions. The authors conclude that physically meaningful solutions exist for line-sources of arbitrary strength and for line-sinks with strength less than 2ν, where ν is the kinematic viscosity. For line sinks with strength greater than this value there are only eigen-solutions. The eigenvalues are the values of the slope of the velocity profile at the wall, and their existence seems to mean that the flow can exist under the present hypotheses only for special Reynolds numbers. The solutions given in detail here are those of polynomial form. Results for these are plotted and discussed. The eigen-solution case is also treated.

In an addendum, some remarks are made about an earlier investigation of the same problem, in the boundary-layer approximation, by Wijngaarden [Nederl. Akad. Wetensch., Proc. **45** (1942), 269–275; MR **5**, 247.] The present authors point out that there is qualitative agreement between their velocity profiles and Wijngaarden's for the case of source-flow, and, moreover, that the earlier author encountered a similar situation for sink-flows. There is almost precise agreement between the two investigations as regards the (dimensionless) eigenvalue of the profile slope for large Reynolds numbers.

W. R. Sears (Ithaca, N.Y.).

Hasimoto, Hidenori. A sphere theorem on the Stokes equation for axisymmetric viscous flow. J. Phys. Soc. Japan **11** (1956), 793–797.

Der Verfasser leitet eine allgemeine Beziehung für die Stromfunktion einer Kugel in achsensymmetrischer, inkompressibler, zäher Strömung her, wobei die Voraussetzung gemacht wird, dass die Geschwindigkeit so klein ist, dass die Trägheitsglieder in der Navier-Stokes'schen Differentialgleichung vernachlässigt werden können. Wird die Kugel $r=a$ in die durch die Stromfunktion $\psi(r, \theta)$ (r, θ, φ Polarkoordinaten) beschriebene Grundströmung gesetzt, dann ist die Stromfunktion der Kugelströmung $\psi(r, \theta)+\tilde{\psi}(r, \theta)$ wobei sich $\tilde{\psi}(r, \theta)$ errechnet aus

$$\tilde{\psi}(r, \theta)=\tilde{\psi}_1(r, \theta)+a^2\psi_2(r, \theta),$$

$$\psi^2=\left(a-\frac{r^2}{a}\right)\frac{\partial}{\partial R}\varphi_1(R,\theta)-\left(\frac{3}{2}\frac{r}{a}-\frac{1}{2}\frac{r^3}{a^3}\right)\varphi_1(R, \theta),$$

$$\psi_2=\left(a-\frac{r^2}{a}\right)\frac{\partial}{\partial R}\varphi_2(R, \theta)-\left(\frac{1}{2}\frac{r}{a}+\frac{1}{2}\frac{r^3}{a^3}\right)\varphi_2(R, \theta),$$

$$\varphi_2=-\frac{\sin\theta}{4r^4}\int_0^r r^4\omega(r, \theta)dr \quad \varphi_1=\psi(r, \theta)-r^2\varphi_2(r, \theta),$$

wobei ω die Drehung der Grundströmung und $R=a^2/r$ bedeutet. Diese allgemeine Beziehung wird auf die Kugelströmung in einer Parallelströmung, in der Strömung einer punktförmigen Quelle und eine Strömung $u=\alpha r^2\sin^2\theta$, $v=0$ angewandt. *L. Speidel.*

Pearcey, T.; and McHugh, B. Calculation of viscous flow around spheres at low Reynolds numbers. Phil. Mag. (7) **46** (1955), 783–794.

Detailed computations have been carried out, on the basis of Oseen's approximation to the hydrodynamic equations of motion, of the flow pattern around a uniformly translated sphere at Reynolds numbers, R, of 1, 4 and 10. A strongly marked wake is apparent even at the lowest Reynolds number. The boundary of the wake becomes more marked as R increases and is bounded by a thin layer in which the fluid motions are particularly small and of rapidly changing direction. No vortex is attached to the rear of the sphere even at $R=10$, although the presence of a 'boundary layer' to the front is well indicated. (Author's abstract.) *W. J. Nemerever.*

Stojanović, Dragutin. Temperaturgrenzschichten in dreidimensionalen Strömungen. Z. Angew. Math. Mech. Sonderheft (**1956**), S30–S31.

Es wird die Temperaturgrenzschicht der inkompressiblen, laminaren Staupunktströmung für den allgemeinen dreidimensionalen Fall behandelt, bei dem die Geschwindigkeitskomponenten der Aussenströmung sind

$$U=a\xi, \quad V=b\eta$$

(ξ, η, ζ krummlinige, rechtwinklige Koordinaten, ξ senkrecht zur Körperoberfläche). Die Grenzschichtgleichungen der Strömungsgrenzschicht werden entsprechend zu dem Lösungsansatz bei der ebenen und der rotationssymmetrischen Staupunktströmung in zwei gewöhnliche Differentialgleichungen übergeführt, wobei $c=b/a$ als Parameter auftritt. Die Energiegleichung wird durch die Transformationen

$$T=x^ny^mv(z),$$

$$T=x^nv_1(z)+y^nv_2(z)$$

(x, y, z cartesische Koordinaten, z senkrecht zur Körperoberfläche, T Temperatur), die den Verlauf der Wandtemperatur nach einem Potenzgesetz voraussetzen, bei Vernachlässigung der Dissipationsglieder in eine gewöhnliche Differentialgleichung übergeführt. Bei Berücksichtigung der Dissipationsglieder ist dies nur für den Sonderfall $n=m=2$ bei dem Transformationsansatz möglich. *L. Speidel* (Mülheim/Ruhr).

Stuart, J. T. On the effects of the Reynolds stress on hydrodynamic stability. Z. Angew. Math. Mech. Sonderheft (**1956**), S32–S38.

This paper gives a general discussion of the problem of finite amplitude disturbances of the plane laminar flow between two parallel walls. In the author's formulation, which is restricted to two dimensions and periodicity in the flow direction x, the total flow (u, w, p) at any instant is divided into a mean part and a fluctuating part, e.g. $u(x, z, t)=\bar{u}(z, t)+u'(x, z, t)$, where mean values with respect to x are implied and z is in the direction normal to the walls. Thus, the mean quantities are also functions of the time and satisfy the Reynolds equations, in which Reynolds stress terms appear. The latter also appear in the corresponding equations for the fluctuations.

On this basis, the energy transfer between the mean and fluctuating motion is discussed, the role of the Reynolds stresses being emphasized. Next is given a summary of the earlier results of Meksyn and Stuart [Proc. Roy. Soc. London. Ser. A. **208** (1951), 517–526; MR **13**, 792] on the critical Reynolds number for a finite

disturbance (the lowest Reynolds number for which a neutral oscillation of given finite amplitude can exist). This is generally lower than the instability limit for infinitesimal disturbances. At Reynolds numbers above the latter, an infinitesimal disturbance may grow until it develops into a neutral oscillation of finite amplitude. An illustration of this property is presented using Burgers' one-dimensional model equation [Advances in Appl. Mech., Academic Press, New York, 1948, pp. 171–199; MR 10, 270].

{The reviewer notes that another formulation, not mentioned by the author, is sometimes used. There is a possibility of misunderstanding in this respect. The total flow can also be divided into a basic (or undisturbed) part (u_0, w_0, p_0) satisfying the steady Navier-Stokes equations and a disturbance (u_1, w_1, p_1), e.g. $u(x, z, t)=u_0(z)+u_1(x, z, t)$. The subscripts 0 and 1 are often replaced by an overbar and a prime. The appropriate equations for the disturbance quantities are similar to the author's fluctuation equations, except for the absence of terms analogous to the Reynolds stress terms. The two formulations should be equivalent ultimately.}

D. W. Dunn (Baltimore, Md.).

★**Zysina-Molozhen, L. M. On the character of the transition from the laminar to the turbulent flow region in the boundary layer.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 12 pp.

Translated from Ž. Teh. Fiz. **25**, no. 7, 1956, 1280–1287.

★**Zysina-Molozhen, L. M. Certain quantitative characteristics of the transition from laminar to turbulent flow in the boundary layer.** Translated by Morris D. Friedman, 572 California St. Newtonville 60, Mass., 1956. 11 pp.

Translated from Ž. Teh. Fiz. **25**, no. 7, 1955, 1288–1296.

Keller, J. B. Spherical, cylindrical and one-dimensional gas flows. Quart. Appl. Math. **14** (1956), 171–184.

The equations of compressible fluid flow are formulated in terms of Lagangrian variables for the cases of spherical, cylindrical or planar flow of a polytropic gas. They can be resolved into a non-linear second order partial differential equation. A solution by separation of variables is obtained and is found to depend on an arbitrary function connected with the entropy state of the gas. By specifying this function certain boundary value problems can be solved for isentropic and non-isentropic flows. In particular the author considers the free expansion of a gas into a vacuum, a shock (strong enough so that special conditions apply across it) moving into a medium of variable density (this yields in special cases solutions like those of G. I. Taylor's strong shock theory), and a finite shock moving into a variable medium. *Hirsh Cohen.*

Schubert, H.; und Schincke, E. Zum Konturproblem der Hodographenmethode im Unterschall. Z. Angew. Math. Mech. **36** (1956), 307–309.

In a previous paper, a certain approximation to the pressure-density relation was suggested [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **101** (1955), no. 6; MR **17**, 312]. This has the virtue of reducing the differential equation for the stream function in the hodograph plane to a very simple form. Here the method is applied to the solution of the "direct problem" of subsonic, plane, steady, irrotational flow about a doubly symmetrical cylinder without circulation. Details, however, are still to be published, in the same series of Berichte. *W. R. Sears* (Ithaca, N.Y.).

Keune, Friedrich; und Schmidt, Werner. Zur Berechnung von Strömung und Wellenwiderstand bei Flügel-Rumpf-Kombinationen in der linearen Überschallströmung. Z. Angew. Math. Mech. **36** (1956), 301–303.

Dans cette courte note, les auteurs comparent les résultats donnés par l'application de la théorie linéaire des corps élancés à un même obstacle, pour deux nombres de Mach différents. *P. Germain* (Paris).

Miles, John W. On the aerodynamic instability of thin panels. J. Aero. Sci. 23 (1956), 771–780.

In his usual elegant style, author treats the linearized problem of travelling waves along a two-dimensional (infinite span) thin panel immersed in a compressible (potential flow) fluid stream; the panel is an initially plane plate-membrane, is oriented in the stream direction and is of infinite length (chord). The panel is unsupported along its chord. Principal interest resides in the boundaries separating stable and unstable wave propagation (i.e., waves which decay or grow in amplitude, respectively).

For this problem, the velocity potential for the fluid has a particularly simple form, as does the characteristic equation. By means of an analysis based on the Cauchy-Nyquist diagram, the stability boundaries are delineated, both for incompressible flow and for all values of the flow Mach number.

Based on his very complete results for the travelling wave problem, the author next attempts to relate this information to the stability problems for panels periodically supported, and for supported panels of finite length. As the author points out, these discussions are illuminating rather than definitive, particularly in the case of the finite-length panel. However, a number of interesting, apparent contradictions appear between the present results and earlier researches. Some question is raised regarding earlier analyses employing a limited number of Lagrangian coordinates; apparently, considerable care must be exercised if such an approach is to be adequate in describing this new type of flutter phenomenon.

M. Goland (San Antonio, Tex.).

Miles, John W. The compressible flow past an oscillating airfoil in a wind tunnel. J. Aero. Sci. 23 (1956), 671–678.

The problem described by the title is attacked in the linear small-perturbation approximation, for both subsonic and supersonic plane flow. For supersonic flow the method of Laplace transforms is used; the results can be recognized as the superposition of the effects of image airfoils in the tunnel walls. Alternatively, they can be interpreted in terms of guided waves, i.e. waves propagating within the wind-tunnel walls. For subsonic flow, Fourier transforms are used and an integral equation is obtained which is equivalent to that found by Runyan and Watkins [NACA Rep. no. **1150** (1953); MR **16**, 194] by a different method. Here the kernel is shown to represent a wake carried downstream by the flow and a system of damped and undamped guided waves. In the closing section, the dynamic-stability derivatives associated with pitching oscillation are determined for the supersonic case, to first order in the reduced frequency. The results predict an extension of the range of Mach numbers over which unstable single-degree-of-freedom oscillations may occur. *W. R. Sears* (Ithaca, N.Y.).

Roy, Maurice. Théorie du profil d'aile à jet. C. R. Acad. Sci. Paris 242 (1956), 3017–3021.

Une expression générale, valable même en fluide compressible mais sans ondes de choc, est donnée pour l'action résultante d'une telle aile, captant du fluide ambiant et émettant un jet laminaire.

Une théorie approchée, rattachable aux approximations de Prandtl, est donnée en écoulement incompressible pour l'aile à simple jet. (Résumé de l'auteur.)

J. B. Serrin (Minneapolis, Minn.).

Li, Ta. Aerodynamic influence coefficients for an oscillating finite thin wing in supersonic flow. J. Aero. Sci. 23 (1956), 613–622.

The author expounds a numerical method for the calculation of the pressure distribution on an oscillating wing of arbitrary plan form in supersonic flow. The method is based on the division of the wing into a number of small rectangles over which the variables of the flow (e.g. the normal velocity) are supposed to be approximately constant. Similar methods have been worked out by S. Pines and J. Dugundji, and by B. Etkin. The particular feature of the present method is that the diagonals of the rectangles are parallel to the Mach lines. For some cases a lift-cancellation technique is suggested, which requires the solution of a system of linear equations. Good agreement with the results of exact linearised theory is obtained for the two-dimensional case and it is implied that further work is in hand to develop and test the method for the general three-dimensional case for which it is chiefly designed. *A. Robinson* (Toronto, Ont.).

Bruniak, R. Über die Ablösung der Grenzschicht beim Verdichtungsstoss. Österreich. Ing.-Arch. 10 (1956), 129–133.

It is proposed to use the Kármán-Pohlhausen approximations to estimate whether a normal shock wave causes the laminar boundary layer at a plane wall to separate. In the simplified model assumed here, the momentum loss of the boundary layer is due to the shock pressure-rise; this gives a relation between the values of the profile parameter λ before and behind the shock for any Mach number. *W. R. Sears* (Ithaca, N.Y.).

Uchida, Shigeo; and Yasuhara, Michiru. The rotational field behind a curved shock wave calculated by the method of flux analysis. J. Aero. Sci. 23 (1956), 830–845.

The method of "flux analysis" effectively reduces the partial differential equation for the rotational flow to an ordinary differential equation. The flow pattern and the form of the shock wave are first assumed, and corrected by a method of successive approximation which involves integration in one direction only. The paper gives detailed description of the method, comparison with other methods including the use of characteristics, and two worked examples whose solutions are found to be in fair agreement with experimental results. *L. Fox* (Berkeley, Calif.).

Häfele, Wolf. Zur analytischen Behandlung ebener, starker, instationärer Stosswellen. Z. Naturf. 10a (1955), 1006–1016.

The methods used by G. Guderley in his well-known discussion of spherical and cylindrical shockwaves [Luftfahrtforschung 19 (1942), 302–311; MR 5, 19] are applied to plane progressing waves [cf. Courant and Friedrichs, Supersonic flow and shockwaves, Interscience, New York, 1948; MR 10, 637] following a strong shockwave. The author finds a unique solution satisfying all the requirements of a self-maintained shockwave. It is given explicitly for a ratio of specific heats $\gamma=1\cdot4$ and exhibits a linear velocity distribution behind the shockwave in this case; the solution has been calculated numerically for $\gamma=1\cdot1$, 5/3 and 2·8. This solution is the one to which the various initial value problems solved numerically by K. Hain and S. v. Hoerner [Z. Naturf. 9a (1954), 993–1004; MR 17, 101] were found to tend with increasing time.

D. C. Pack (Glasgow).

v. Hoerner, Sebastian. Lösungen der hydrodynamischen Gleichungen mit linearem Verlauf der Geschwindigkeit. Z. Naturf. 10a (1955), 687–692.

It is shown that there are three general types of solution of the equations of unsteady plane gas flows, and three limiting cases, which exhibit a linear distribution of velocity. These distributions can in general not represent flow behind a strong shockwave, but for a ratio of specific heats $\gamma=7/5$ an exception has to be made [cf. the paper reviewed above]. For $\gamma=5/3$ they represent a good approximation to Häfele's standard solution back to the point of zero velocity. Another example of a linear velocity distribution is given by J. M. Burgers' interstellar gas cloud [K. Ned. Akad. v. Wet. 49 (1946), 588–607].

D. C. Pack (Glasgow).

Meyer, F. Zur Darstellung starker Stossfronten durch Homologie-Lösungen. Z. Naturf. 10a (1955), 693–697.

The development of the flow behind a strong plane shockwave is expressed by means of a series of powers of the distance behind the front. By assuming that there is little variation of the highest coefficient retained in the discussion and that higher coefficients are unimportant, the author shows that the flows behind large classes of plane shockwaves tend with time towards "progressing waves" and that under further restrictions on this coefficient, sufficient but perhaps not necessary, they tend towards the particular progressing wave found in the paper reviewed second above. *D. C. Pack* (Glasgow).

Häfele, Wolf. Über die Stabilität des Stosswellentypus aus der Klasse der Homologie-Lösungen. Z. Naturf. 10a (1955), 1017–1027.

The flow behind a strong shockwave involving only one space coordinate x and the time t is considered. The velocity is expressed in the form

$$u=n(t)t^{n(t)-1}\phi v(\phi, n(t))$$

where $\phi=x/\int_0^t n(\tau)\tau^{n(\tau)-1}d\tau$. All other variables are expressed in similar terms such that $n(t)=$constant yields the "progressing waves". These substitutions, which entail no loss of generality, lead to equations of motion which may be put in such a form that the progressing waves are those for which the terms on the right-hand sides vanish identically. The author considers plane flows approximating to the progressing wave $n=n_0$ (say) which describes the flow behind a self-maintained plane shockwave [see the third preceding review]. These neighbouring flows ("N-solutions") are required to be such that they do not lead to limiting lines behind the shock front. It is shown that as a consequence of this condition the values of $n(t)$ corresponding to N-solutions satisfy the inequality $dn/dt \lesseqgtr 0$ according as $n \lesseqgtr n_0$. In this sense the self-maintained shockwave is stable. *D. C. Pack* (Glasgow).

Häfele, Wolf. Über die Stabilität des Guderleyschen kugeligen Verdichtungsstosses. Z. Naturf. **11a** (1956), 183–186.

The author applies the method developed in the paper reviewed above to show that in the neighbourhood of the origin Guderley's spherical shockwaves [Luftfahrtforschung **19** (1942), 302–311; MR **5**, 19] are stable with respect to such disturbances as are permitted for the application of the method. *D. C. Pack* (Glasgow).

Zelenskiĭ, I. E. On the drag of bodies immersed in a gaseous flow with supersonic velocity. Har'kov. Gos. Univ. Uč. Zap. **29**=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **21** (1949), 11–22. (Russian)

Suppose the region of uniform supersonic flow parallel to the x-axis ahead of a body is terminated at a shock S_1 which may be followed by additional shocks S_i farther downstream. Calculate the flow of momentum through a control surface consisting of S_1 and the plane $x=\infty$. The author shows that this yields for the head drag of the body the value

$$W=(\rho_0v_0^2+p_0)\iint_{S_1}\{-1+\exp(\sum_i [H]/R)\}dydz,$$

where subscripts 0 refer to the undisturbed flow and $\sum_i [H]$ is the total increase of specific entropy experienced by a particle in its passage through the entire system of shocks. *J. Giese* (Aberdeen, Md.).

★**Chernyi, G. G. One-dimensional unsteady motion of a perfect gas with a strong shock wave.** Translated by Morris D. Friedman, 572 California Street, Newtonville 60, Mass., 1956. 6 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **107**, (1956), no. 5, 657–660.

★**Chernyi, G. G. Hypersonic gas flow around a body.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **107**, (1956), no. 2, 221–224.

Holt, Maurice; and Blackie, John. Experiments on circular cones at yaw in supersonic flow. J. Aero. Sci. **23** (1956), 931–936.

The experiments reported here are related to the theoretical studies of Stone [J. Math. Phys. **27** (1948), 67–81; and **30** (1952), 200–213; MR **9**, 544; **13**, 702], Kopal et al. [Mass. Inst. Tech., Dept. Elec. Engrg., Center of Analysis, Tech. Rep. No. **5** (1949); MR **11**, 65] and Ferri [NACA Tech. Note no. **2236** (1950); NACA Rep. no. **1045** (1951); MR **12**, 875; **14**, 331]. Measured pressure distributions and lift and drag coefficients are compared with theoretical values. In the discussion of the results certain shortcomings of the first-order theory of Stone, corrected by Ferri, are pointed out. {The reviewer believes that some of the nomenclature of the present paper is unfortunate, since the "entropy correction" to the pressure is actually something else. This is pointed out in the closing remarks, with credit to Professor Ferri, but the nomenclature was not corrected.} *W. R. Sears.*

Todeschini, Bartolomeo. Correnti ipersoniche rotazionali piane ottenute perturbando correnti rettilinee. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 979–989.

Let Ψ be Crocco's stream function defined by $\Psi_y=u(1-w^2)^{1/(k-1)}$, $\Psi_x=-v(1-w^2)^{1/(k-1)}$, where $w^2=u^2+v^2$ for velocity components $(2H)^{\frac12}u$, $(2H)^{\frac12}v$ and stagnation enthalpy H. Then for some Q, characteristic of the flow considered, H and the specific entropy S satisfy $\log(S^{1/k}/H)=Q(\Psi)$. The author considers flows with $\Psi=\overline{\Psi}+\psi$, where $\overline{\Psi}(y)$ corresponds to a shear flow with prescribed distribution $m(y)=2\bar{u}^2/(k-1)(1-\bar{u}^2)$ of the square of the "reduced" Mach number for $\bar{u}(y)$. For $m(y)=A+\beta y$ with constant A and β the reduced velocity $\bar{u}(y)$, 0 is rotational if $\beta\neq0$. If in the partial differential equation for Ψ only first order terms in ψ and its derivatives are retained, ψ is independent of β. If second order terms are also kept, some coefficients depend on β in the resulting quasi-linear equation. The author exhibits an approximate integral ψ of this equation in terms of two arbitrary functions of $y\pm x(A-1)^{-\frac12}$ and their derivatives. For the perturbed flow past a bump $y=\epsilon g(x)$ on the x-axis, or one surface of an airfoil, the pressure increment includes the customary terms involving $g'(x)$, and if $\beta\neq0$ a term proportional to $g(x)$. *J. Giese.*

Frankl', F. I. Subsonic flow about a profile with a supersonic zone. Prikl. Mat. Meh. **20** (1956), 196–202. (Russian)

The author partially formulates a boundary value problem whose solution is intended to lead to an approximation to a symmetrical flow about some profile with cusped leading and trailing edges and with local supersonic zones terminated by straight normal shocks. Variations in entropy behind shocks and discontinuities at the slipstreams off the outboard ends of the shocks are neglected to permit approximation by irrotational flows. The stream function ψ satisfies the Tricomi equation $\eta\partial^2\psi/\partial\theta^2+\partial^2\psi/\partial\eta^2=0$, where η is a distorted speed of flow and θ is the inclination of the velocity. Only the upper half of the symmetrical flow field need be considered. $\partial\psi/\partial\theta=0$ and $\psi(\theta_A,\eta)=\psi(\theta_A,-\eta)$ on the shock, whose image is approximately a segment HAB of $\theta=\theta_A$ bisected at A by the sonic line $\eta=0$. Next $\psi=0$ on CDE, the image of the line of symmetry, where $\theta=0$, while D corresponds to the velocity at infinity, and C and E are not stagnation points. Also $\psi=0$ on the subsonic parts of the profile, whose images BC and EF are apparently to be chosen arbitrarily. By analogy with the hodograph representation of incompressible flow about a circular cylinder, it is assumed that near D we have $\psi=\rho^{-\frac12}\sin\frac12 t+O(\rho^{\frac12})$, where $\theta=\rho\sin t$, $\eta^{1\cdot5}-\eta_D{}^{1\cdot5}=1.5\rho\cos t$. Let G be the intersection of the sonic line and the characteristic from H. On the sonic segment FG $\psi(\theta,0)=f(\theta)$ is assumed given. It is not known how to choose f so that the image of the streamline $\psi=0$ from F will pass through H to guarantee the formation of a profile. However the author does show that the assumed type of singularity at D actually assures that in the physical plane the chord CE of the desired symmetrical airfoil is parallel to the flow at infinity. Similar discussions are given for extensions to circulatory flows and to boundary value problems for the unsimplified hodograph equations.

J. Giese (Aberdeen, Md.).

Persen, Leif N. Einiges über die Grundlage der Berechnung von Wasserschlössern. Z. Angew. Math. Mech. **36** (1956), 305–306.

Heinrich, G. Gasdynamische Wirkungen von Staublawinen. Z. Angew. Math. Mech. **36** (1956), 298–300.

Barenblatt, G. I.; and Višik, M. I. On finite velocity of propagation in problems of non-stationary filtration of a liquid or gas. Prikl. Mat. Meh. **20** (1956), 411–417. (Russian)

Consider the case of filtration of water in the soil by means of plane waves. The pressure of filtrating water is given by Boussinesq's equation, containing a constant which defines the natural characteristic filtration property of the soil. The authors point out that the analogous equation appears for the pressure in a nonstationary isothermic filtration process of a gas. The pressure of the soil water corresponds to a certain pressure function of the gas. Due to this analogy it is possible to transfer the results obtained for one phenomenon onto the other. With this analogy in mind, the authors consider the case of soil water filtration. In the first part they discuss the possible solution of the fundamental equation for the fluid filtration including such particular items like the behavior of the solution at t equal to zero or approaching infinity, etc. In the second part they discuss most characteristic aspects of the velocity of filtration. In the third part they give the equation for the nonstationary filtration process of a gas, pointing out the features analogous to the soil water filtration process. The paper is limited to general considerations with no particular examples. *M. Z. Krzywoblocki.*

Licher, R. M. Reduction of drag due to lift in supersonic flight by distributing lift along a fuselage. J. Aero. Sci. **23** (1956), 1037–1043.

Mihailov, G. D. Distortion and interaction of acoustic waves of finite amplitude in a viscous medium. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 68–71. (Russian)

This paper treats the propagation of sound waves in one dimension through a viscous medium from (1) a source excited harmonically at one frequency ω_1 and (2) a source excited at two different frequencies ω_1 and ω_2. The equation of state is assumed in the form $\partial p/\partial x = c^2(\rho)\partial\rho/\partial x$. Following Eckart [Phys. Rev. (2) **73** (1948), 68–76] the density ρ, sound pressure p, particle velocity, and coefficient $c^2(\rho)$ are expressed as power series in a perturbation parameter. An expression for the second-order density ρ_2 in terms of the zeroth- and first-order variables is assumed, and on this basis the solution p_2 of the second-order equations is obtained for problem (1). For slightly viscous media or great distances from the source this reduces to the form

$$p_2 = k_0 x \sin[2\omega_1(t-x/c_0)+\pi],$$

and for the opposite limiting case

$$p_2 = k_0 e^{-2\alpha_1 x} \sin[2\omega_1(t-x/c_0)+\pi].$$

For problem (2) only the "combination waves" involving the sum and difference $(\omega_1 \pm \omega_2)$ frequencies are considered. The author concludes that the theory predicts spatial maxima in the intensity of the harmonics as well as of the combination waves, and that for a given frequency the location of the maxima is determined by the viscosity. *R. N. Goss* (San Diego, Calif.).

Owczarek, J. A. Theoretical investigation of the influence of viscous friction on a plane wave of finite amplitude in a compressible fluid. Quart. J. Mech. Appl. Math. **9** (1956), 143–163.

This paper deals with the theory of the subsonic motion of plane waves of finite amplitude in gases with viscous effects, without reflections and interactions. The method of characteristics as applied to the problem furnishes equations suitable for step-by-step solutions. An extension of this method allows deduction of analytical expressions for the change of parameters of state of a wave, due to friction; these expressions can be integrated numerically. An approximate method of solution of these equations is indicated. *Author's summary.*

Tyutekin, V. V. Propagation of elastic waves in a medium with cylindrical canals. Akust. Ž. **2** (1956), 291–301. (Russian)

A plane acoustic wave is normally incident on the uniformly perforated surface of a semi-infinite elastic medium. Perforations are infinitely long circular canals with axes perpendicular to the surface. For simplicity, characteristic equation of wave propagation through medium is derived for cylindrical elastic element instead of hexagonal element surrounding a canal. Solution of this equation is finally developed for a "resin-like" material which has only volume elasticity, like water, and a shear modulus. The presence of the canal is found to convert volume deformations in the solid resin to shear deformations at the canal surface. The wave number of this "micro-nonhomogeneous" medium was found to depend not on the number of canals per unit of surface, but only on the square of the "perforation coefficient" (i.e., square of ratio of canal diameter to canal spacing). *W. W. Soroka* (Berkeley, Calif.).

Voit, S. S. Reflection of sound waves from an oscillating plane. Trudy Moskov. Mat. Obšč. **5** (1956), 81–88. (Russian)

The symmetrical and antisymmetrical solutions of $\phi_{tt} = c^2\phi_{xx}$ for the boundary condition $(\phi_x)_{x=b\cos\omega t} = -b\omega \sin \omega t$ are obtained as Fourier expansions. They correspond to plane sound waves excited and/or reflected by a vibrating boundary; frequency auto-modulation is involved, and there is formal identity with the classical Kepler problem. The more general problem of standing waves of frequency $\sigma \neq \omega$ is similarly solved; determination of the coefficients of the combination waves with $n(\omega/\sigma) \pm 1$ likewise involves the Kepler equation. Analyticity of the solution is predicted on the condition $\omega b < c$ (velocity amplitude of boundary < sound velocity); otherwise discontinuities (surging) obtain: multiple intersection of the characteristics of the wave equation with the line corresponding to the boundary condition. A graphical method readily yields sketches of the complex wave forms. *H. G. Baerwald* (Cleveland, Ohio).

Brehovskih, L. M. Propagation of sound in inhomogeneous media. Akust. Ž. **2** (1956), 235–243. (Russian)

A qualitative description of sound propagation phenomena in media having layer-type inhomogeneity and in media having statistical inhomogeneity. Sound channels, shadow zones, temperature- and wind-gradient refraction effects, diffraction, diffusion, scattering due to atmospheric turbulence and ocean inhomogeneities are described. Reference is made to mathematical researches of several investigators in this field. *W. W. Soroka.*

See also: Yanenko, p. 214; Maslennikova, p. 215; Formica, p. 236; Garbsch, p. 236; Zierep, p. 237; Hall, p. 237; Press and Tukey, p. 245; Peterlin, p. 257; Longuet-Higgins, p. 263; Thompson, p. 264.

Optics, Electromagnetic Theory, Circuits

Kagan, R. L.; and Yudin, M. I. Approximate solution of the equation of dispersion of light. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 968–975. (Russian)

"The authors consider the problem of dispersion of light in a plane layer which is illuminated from above by a parallel bundle of light and bounded from below by a surface with known albedo. An approximate solution is derived which for moderate values of the optical thickness of the layer produces better results than the formulas of Schwarzschild and of Eddington" (Author's summary). The mathematical problem reduces to a system of nonlinear integro-differential equations. The approximate method of the paper consists in replacing the terms which make for non-linearity by judiciously chosen constants. No analysis of the error is given, but there are numerical comparisons with earlier approximate results by Chandrasekhar [Radiative transfer, Oxford, 1950; MR **13**, 136] and others. *P. Henrici* (Los Angeles, Calif.).

Harris, Joseph D. On the diffusion of ions in membranes. Bull. Math. Biophys. **18** (1956), 255–261.

Djurić, Jovan S. On multiple-valued functions and Sommerfeld's surface wave. Bull. Acad. Serbe Sci. Cl. Sci. Tech. (N.S.) **13** (1955), no. 4, 115–119.

Comments on this much disputed subject. Criticism of a series of papers by Kahan and Eckart [see MR **9**, 637; **11**, 143; **12**, 224; see also Epstein, MR **9**, 126]. *C. J. Bouwkamp* (Eindhoven).

★**Dolukhanov, M. P. Radiowave propagation. Chapter 4. Long Radiowave Propagation.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 14 pp.

Translated from Gosudar. Izdat. Liter. po Voprosam Svyazi i Radio, Moskva, 1952.

★**Dolukhanov, M. P. Radiowave propagation. Section 7.7. Effect of meteorological regions in the lower layers of the atmosphere on ultra-short wave propagation conditions.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 33 pp.

Translated from Gosudar. Izdat. Liter. po Voprosam Svyazi i Radio, Moskva, 1952, 436–462.

★**Ostianu, V. M. On the synthesis of switching circuits with selector switches.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 7 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 827–830.

Jauquet, C. Lancement d'une onde de surface transverse magnétique sur un cylindre diélectrique. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 802–812.

Consider an electromagnetic horn opening to the right, the mouth (of radius b) of which lies in a perfectly conducting plane S. To the right of S and coaxial with the horn extends a right circular cylinder of radius $a<b$, where a is also the radius of the conducting element in the horn. To find the field in the half-space to the right of S, the author studies the equivalent problem of the field of an infinitely thin plane ring of magnetic current, coaxial with the horn and of radius s, finally summing the contributions from $s=a$ to $s=b$. From a study of the singular points of the resulting integral, he is able to show that this field includes a wave which predominates near the surface of the cylinder and far from the source. *R. N. Goss* (San Diego, Calif.).

Bunkin, F. V. The noises of the cyclic transmagnetisation of ferromagnetic materials. Ž. Tehn. Fiz. **26** (1956), 1790–1798. (Russian)

Rubinow, S. I.; and Wu, T. T. First correction to the geometric-optics scattering cross section from cylinders and spheres. J. Appl. Phys. **27** (1956), 1032–1039.

The authors discuss the total scattering cross section of plane waves impinging on cylinders and spheres. Both Dirichlet and Neumann boundary conditions are considered. The aim is to derive the first correction term to the geometrical optical cross section.

The method is the following: The cross section is written in terms of the phase shifts using Hankel functions. The Hankel functions are replaced by their asymptotic expansions, and the summation over the phase shifts is approximated by an integration. This integral can be readily evaluated numerically.

The results are the following: In each case the first correction term is proportional to $(ka)^{-\frac{2}{3}}$; where k is the wave number, and a is the radius of the cylinder or sphere. The correction increases the scattering cross section for the Dirichlet boundary condition, while decreasing it for the Neumann boundary condition. Moreover, the first correction to the total scattering cross section of a conducting sphere by a plane electromagnetic wave is the average of the first corrections to the Dirichlet and Neumann cases for the sphere. (The authors point out that physically the first correction arises from the grazing rays.)

A numerical comparison is made between the exact scattering cross section and its two-term asymptotic expansion. The agreement is very good. *N. L. Balazs.*

Kovács, R.; and Solymár, L. Theory of aperture aerials based on the properties of entire functions of the exponential type. Acta Phys. Acad. Sci. Hungar. **6** (1956), 161–184. (Russian summary)

The authors use properties of entire functions of exponential type to discuss a number of questions about aperture aerials. A prescribed radiation pattern cannot in general be realized exactly, but can in arbitrarily close approximation; the feasibility of doing this in a practical way is discussed. A bound is obtained for the quality in terms of the gain. The mathematical appendix contains statements and proofs of the relevant theorems on entire functions. All the theorems seem to be already available in the mathematical literature. *R. P. Boas, Jr.*

Peterlin, Anton. The effect of a transverse electric field on the intrinsic viscosity of a suspension of dipole ellipsoids. Slovenska Akad. Znan. Umet. Razred Mat. Fiz. Tehn. Vede. Ser. A. Razprave **7** (1956), no. 1, 18 pp. (Slovene and English)

Author's theory gives increase of viscosity η in terms of amplitude E and frequency ν of the electric field as

$$\Delta\eta/\eta=E^2(A+B\nu^2),$$

where A and B are constants deducible from material properties. Experiment verifies the form of the relationship but gives values of the constants different from those predicted. {Reviewer found paper difficult to follow.} *D. R. Bland* (London).

Collins, W. D. Note on the two-dimensional theory of anisotropic dielectrics. Mathematika 3 (1956), 63–68.

The electrostatic field in the interior of a homogeneous anisotropic dielectric is found by a modification of the method of complex potentials. The governing equations are $\mathbf{E}=-\operatorname{grad}\phi$, $\operatorname{div}\mathbf{D}=0$, and $\mathbf{D}=(k_{ij})\mathbf{E}$, where (k_{ij}) $(i, j=1, 2)$ is symmetric, positive definite. The potential ϕ satisfies the second order complex equation

$$s\frac{\partial^2\phi}{\partial t^2}+2t\frac{\partial^2\phi}{\partial z\partial \bar{z}}+\bar{s}\frac{\partial^2\phi}{\partial \bar{z}^2}=0,$$

where $s=k_{11}-k_{22}+2ik_{12}$, $t=k_{11}+k_{22}$.

The general solution is obtained in the form

$$2\phi=\Omega_1(z+\lambda\bar{z})+\bar{\Omega}_1(\bar{z}+\bar{\lambda}z),$$

where Ω_1 is arbitrary analytic and $\bar{s}\lambda_1=-t+\sqrt{(t^2-s\bar{s})}$. The field is then obtained in the form

$$E_x+iE_y=-2\frac{\partial\phi}{\partial z}=-\lambda_1\Omega_1'(z+\lambda\bar{z})-\Omega_1'(\bar{z}+\bar{\lambda}z).$$

The method is then applied to find the field in a dielectric elliptical cylinder surrounded by a uniform vacuum field, a line charge in the exterior of a dielectric half space, a line charge parallel and exterior to a dielectric circular cylinder. *A. A. Blank.*

★ **Westcott, J. H. Driving-point impedance synthesis using maximally lossy elements.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 63–78. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1956.

A simple procedure is given which modifies the synthesis steps in the methods of Foster, Cauer, and Brune. The modified steps tend to give series-resistance to the coils and shunt-resistance to the condensers. *R. J. Duffin.*

Rohleder, Hans. Die Verwendung von Aussagenkalkülen zur Beschreibung elektrischer Schaltungen. Z. Math. Logik Grundlagen Math. 1 (1955), 304–309.

An expository paper indicating how the propositional calculus can be used to analyze and synthesize relay networks. The author prefers (as do many logicians) a description in terms of the propositional calculus rather than one in terms of Boolean algebras.

A recursive definition of series-parallel contact networks is given, and a one-to-one correspondence is established between them and proposition letter formulas containing only disjunction, conjunction and literals. {The reviewer suggests that a natural interpretation of the letters is to consider them to be abbreviations of statements concerning switching-element status, rather than variables.} Equivalent networks are defined and shown to have equivalent logical formulae associated with them.

Another one-to-one correspondence is established between sequences of p.l.f. (one associated with each node of the net) and non-series-parallel networks. {The reviewer suggests that a natural interpretation of the node letters is to consider them to be abbreviations of statements concerning whether the node is at battery (or ground) potential, rather than variables.} It is stated that these sequences of expressions can be manipulated to yield equivalent circuits but no details are given on either optimization or synthesis.

The most general circuit considered is a n-terminal contact network with one terminal at battery potential, and the others connected to individual relay coils which are returned to ground. A sequence of expressions, one for each coil, is then derived. The analysis is thus related to the joint satisfiability of the set of expressions. The author suggests that a three-valued propositional calculus would eliminate the problem of the unstable (unsatisfiable) configuration. No supporting details are adduced. *G. W. Patterson* (Philadelphia, Pa.).

★ **Weber, Ernst. Linear transient analysis. Volume II.** John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. xiv+452 pp. \$10.50.

This is a worthy successor to the first volume [1954; MR 16, 203], which was well received two years ago. It covers essentially the second semester of the basic graduate course on transient analysis offered at PIB. It is self-contained, including most of the material of the first volume at a somewhat less leisurely pace. One of the best known educators in the field, the author understands how to tread the narrow path between strictly elementary approach and inclusion of subject matter calling for advanced mathematical treatment; excellent documentation supplements the presentation in this respect, and some of the problems are designed to whet the appetite to follow up on the sources quoted. For example, the chapter on feed-back amplifiers covers stability theory, but not synthesis, though the developments based on complementary Fourier integrals (Wiener-Lee) and culminating in Bode's and Hansen's work are quoted.

The introductory chapter deals with the Fourier integral and Laplace transform, covered in the last two chapters of Volume I, in preference to the Heaviside operational approach which is mentioned in passing. Chapters 2 to 5 are devoted to four-poles or rather "two-ports": the conventional matrix description, though not including the more recent scattering matrix approach, and a brief section on response to frequency-modulated signals; wave filters: conventional image theory with steady state treatment followed by step function response, plus a brief review of mechanical and thermal analogues; active four-poles, with small signal theory of tubes and transistors, and basic feedback theory, not including control and servomechanisms. The remaining chapters deal with transmission lines, starting with waves on loss- and distortionless lines, infinite, finite and terminated by lumped elements; now inductive lines including the asymptotic approach: semiconvergent treatment, without introduction of the steepest descent method; and general transmission lines. In this last chapter contour integration involving Bessel functions is used to some extent. These and other mathematical tools are developed lucidly when required; there are, however, two short separate appendices setting forth matrices and determinants and the elements of functions of a complex variable.

The book is well printed and provided with diagrams. *H. G. Baerwald* (Cleveland, Ohio).

★ **Kulikovskii, A. A. Comparison of the theories of tube and semiconductor amplifiers and their possible generalization.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 13 pp.

Translated from Radiotehnika 10, no. 11, 1955, pp. 3–11.

★ **Filippov, L. I. Potential interference-stability in the reception of pulse radio signals.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 15 pp.

Translated from Radiotehnika 10, No. 10, 1955, pp. 39–50.

See also: Keller; Kay; and Shmoys, p. 204; Bochenek, p. 213; Moses, p. 216; Lloyd and McMillan, p. 242; Kar, p. 261; Marx and Györgyi, p. 261.

Quantum Mechanics

Bopp, Fritz. Würfel-Brettspiele, deren Steine sich näherungsweise quantenmechanisch bewegen. Z. Naturf. **10a** (1955), 783–789.

In order to illustrate a certain analogy between quantum mechanics and stochastic processes [see also the paper reviewed below], the author considers examples of such processes provided by simple games and shows that in certain cases, for special initial probability distributions, the approach to the stationary distribution is extremely slow and, with neglect of this small damping, the time evolution has an oscillatory character reminiscent of the quantum-mechanical situation. *L. Van Hove.*

Bopp, Fritz. Quantenmechanische und stochastische Prozesse. Z. Naturf. **10a** (1956), 789–793.

This paper can only be understood on the basis of a later paper of the author [see paper reviewed second below]. It deals with a finistic idealization of a quantum-mechanical system and describes it by means of the average values of certain operators F_{mn} defined as special combinations of the density operators in ordinary and momentum space. The crucial point is that the F_{mn} form a complete basis in the linear space of all operators. As a result their average values satisfy with respect to time a linear system of first-order differential equations. It is shown that a modification of these equations in the very high frequency range gives rise to equations describing a stochastic process. This process has a very long relaxation time for the actually observed distributions of average values. *L. Van Hove* (Utrecht).

Bopp, Fritz. Einfaches Beispiel aus der stochastischen Quantenmechanik. Z. Physik **143** (1955), 233–238.

As shown also in the two papers reviewed above it is possible to construct stochastic systems the time evolution of which agrees to very high approximation with the time evolution of a quantum-mechanical system. An example of this analogy is given. *L. Van Hove.*

Bopp, Fritz. Eine Art Phasenraum-Darstellung der Quantenmechanik. Z. Physik **144** (1956), 13–24.

The author studies the same model of quantum-mechanical system as in the paper reviewed second above. Some of the definitions, notations and equations required for the understanding of the paper just mentioned are to be found in the present publication. *L. Van Hove.*

★**Brinkman, H. C. Applications of spinor invariants in atomic physics.** North-Holland Publishing Co., Amsterdam; Interscience Publishers, Inc., New York, 1956. x+74 pp. $3.25.

The author gives a survey of Kramers' technique for using the theory of two component spinor invariants in the calculation of non-relativistic atomic properties. In Chapter I the two component spinors are introduced via the correspondence between complex three dimensional vectors and singular involutions in a complex two dimensional space. By this means a double valued representation of the three dimensional real rotation group by unitary transformations in a complex two dimensional space is explicitly established. The role this representation plays in discussing the transformation properties of the solutions of the Schrödinger equation for a single electron without spin in a central symmetric field of force is discussed. The chapter concludes with a discussion of invariant spinor polynomials and their expression in terms of invariant spinor binomials. Chapter II discusses the classification of wave functions with respect to angular momentum quantum numbers and applies the method of spinor invariants to the calculation of angular momentum matrix elements. Included in this chapter is a detailed discussion of the Pauli theory of electron spin. Chapter III gives a detailed discussion of the application of spinor invariants to the electrostatic interaction in two-electron systems, the spin orbit interaction and the intensities of spectral lines. *A. H. Taub* (Urbana, Ill.).

Risberg, Vidar. On a new method for calculation of scattering phase shifts. Arch. Math. Naturvid. **53** (1956), 1–8.

The author obtains a maximum value for the phase shift in a scattering problem by means of the following procedure. He asks for what values of the wave number k will the wave function has a zero at a particular point which is essentially outside the range of forces, but is otherwise at his discretion. This problem can be solved by the standard Ritz procedure giving a minimum value to the wave number k and a corresponding maximum value for the phase shift. The method is illustrated by a numerical example. *H. Feshbach* (Cambridge, Mass.).

Faddeev, L. D. Uniqueness of solution of the inverse scattering problem. Vestnik Leningrad. Univ. **11** (1956), no. 7, 126–130. (Russian)

The Schrödinger equation $\Delta u+k^2u=q(x)u(x;k\cdot\nu)$ is considered for $u(x;k\cdot\nu)=e^{ik(x,\nu)}+v(x;k\cdot\nu)$, where ν is an arbitrary unit vector and $v=O(1/|x|)$. Moreover

$$v=f(n,\nu,k)e^{ik|x|}/|x|+o(1/|x|)$$

where $n=x/|x|$. It is assumed $f(n,\nu,k)$ is known. Inverting the Schrödinger equation one obtains, for $q(x)=O(|x|^{-2-\epsilon})$

$$u(x;k\cdot\nu)=e^{ik(x,\nu)}-\frac{1}{4\pi}\int\frac{e^{ik|x-y|}}{|x-y|}\,q(y)u(y;k\cdot\nu)\,dy$$

with further restriction on q this leads to

$$\lim_{k(n-\nu)=m;\;k\to\infty} f(n,\nu,k)=-\frac{1}{4\pi}\int q(y)e^{-i(m,y)}dy.$$

An iteration for the formula for u is also considered.
N. Levinson (Cambridge, Mass.).

Klepikov, N. P. Application of the theory of singular integral equations to problems of scattering of particles in an external field. Ž. Eksper. Teoret. Fiz. **30** (1956), 701–706, supplement to **30**, no. 4, 5. (Russian. English summary)

The author argues that the theory of singular integral equations applied to Heitler's equation for the transition amplitude [Quantum theory of radiation, 3rd ed. Oxford, 1954, p. 166] provides an advantageous approach to the problem of computing the scattering of particles in an external potential. After giving a brief derivation of Heitler's equation and specializing it to the case of a central potential, he works out the case of a δ-function potential in detail employing the methods of Mushelišvili [Singular integral equations, Gostehizdat, Moscow-Leningrad, 1946; MR **8**, 586; **15**, 434]. It is emphasized that

neither perturbation nor the Fredholm theory is applicable to this case. *A. S. Wightman* (Princeton, N.J.).

Taylor, J. G. Quantum electrodynamics and Hilbert space theory. Proc. Cambridge Philos. Soc. **52** (1956), 719–733.

The author attempts to rigorize the usual second quantization theory of the free electron-positron field, using the standpoint and the methods of J. M. Cook [Trans. Amer. Math. Soc. **74** (1953), 222–245; MR **14**, 825]. The ordinary construction of hole theory is discarded on the ground that, since it requires consideration of states with infinitely many (negative energy) particles present, it cannot be formulated in the rigoristic Hilbert space description of second quantization. This difficulty is avoided by direct definition of electron and positron states of positive energy. Creation and annihilation operators are introduced, not for one-particle plane wave states as is usually done, but for one-particle states of finite norm. The latter restriction causes great difficulties in the subsequent definition of the field operators, because they cannot be made relativistically covariant within the mathematical framework adopted by the author. On the basis of this fact the author concludes that Hilbert space theory is not a suitable tool for a rigorous formulation of quantum field theory. *L. Van Hove.*

Rayski, J. A discussion on bilocality. Acta Phys. Polon. **15** (1956), 89–109. (Russian summary)

The paper is a readable discussion of the background and motivation of bilocal field theory with particular reference to recent work of the author [Nuovo Cimento (10) **2** (1955), 255–272; MR **17**, 929]. It is set in the form of a dialogue between two brothers who are theoretical physicists. They convince each other without great difficulty that the author's ideas flow naturally out of the basic notions of relativity and quantum mechanics.

A. S. Wightman (Princeton, N.J.).

Rayski, Jerzy. A variational principle for bilocal field theory. Acta Phys. Polon. **15** (1956), 123–127. (Russian summary)

In bilocal field theory, the field operator satisfies a symmetrical set of equations which is larger in number than the number of independent field components. In previous work some of the equations were derived as Euler-Lagrange equations of a variational principle and the rest regarded as constraints [see, e.g., C. Bloch, Danske Vid. Selsk. Mat.-Fys. Medd. **26** (1950), no. 1; MR **12**, 292]. In this note, the author gives a variational principle from which all the equations follow so that they are all treated on an equal footing. *A. S. Wightman.*

Suffczyński, M. Two-centre integrals over the atomic sphere. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 273–277.

The author presents explicit formulas for the Coulomb two-centre integrals $(dd\sigma)$, $(dd\pi)$, and $(dd\delta)$ over the atomic sphere (of the central atom). *A. Erdélyi.*

Lozano, Juan M. On the distribution of the poles of the S-matrix in the optical model of the nucleus. Univ. Nac. Autonoma Mexico. An. Inst. Fis. **1** (1955), 49–54. (Spanish)

The scattering of non-relativistic particles by a square well potential with a constant imaginary part is studied. It is shown that the R-matrix is no longer of the type of an "R-function" of Wigner, because its poles are shifted into the lower half of the complex energy plane. Yet the S-matrix has no singularities in the upper half plane, which would give rise to acausal phenomena. {It should be noted, however, that the absence of singularities is a necessary, but not a sufficient condition for causality.}

N. G. van Kampen (Utrecht).

Seiden, Joseph. Réversibilité et irréversibilité en résonance nucléaire. C. R. Acad. Sci. Paris **243** (1956), 1201–1203.

The author states without proof, that, following Van Hove [Physica **21** (1955), 517–540; MR **17**, 115], irreversibility may be derived quantum mechanically for a spin-lattice system without repeated use of the assumption of random phases. *D. L. Falkoff* (Waltham, Mass.).

Eliason, M. A.; Stogryn, D. E.; and Hirschfelder, J. O. Some molecular collision integrals for point attraction and repulsion potentials. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 546–559.

For a repulsive potential of the form $\Phi=dr^{-\delta}$ ($r=$ distance), let

$$\chi(t)=\pi-2\int\left[1-y^2-\frac{1}{\delta}\left(\frac{y}{t}\right)^{\delta}\right]^{-\frac{1}{2}}dy,$$

integration being extended from $y=0$ to the first positive singularity of the integrand. The collision cross sections for the various transport properties depend on the integral

$$A^{(l)}(\delta)=\int_0^\infty(1-\cos^l\chi)t\,dt.$$

The authors give numerical values of this for $l=1$, 2 and $\delta=2$, 3.

In the case of attractive potentials there is a similar integral, with a slightly different definition of $\chi(t)$, and numerical values are given for $l=1$, 2 and $\delta=2$, 3, 6, and for $l=3$, 4 and $\delta=6$. *A. Erdélyi* (Jerusalem).

Wichmann, Eyvind H.; and Kroll, Norman M. Vacuum polarization in a strong Coulomb field. Phys. Rev. (2) **101** (1956), 843–859.

The authors study the polarization of vacuum in a strong coulomb field. They avoid a perturbation calculation by expressing the polarization charge density in terms of the eigenstates of the Dirac equation in a Coulomb field, and neglect radiative corrections. The sum over the energy eigenstates of the Dirac radial equation is transformed into a contour integral of the corresponding Green's function. The properties of this function are investigated. The Laplace transform of the polarization charge density presents ambiguities which are removed by regularization. The first order term of a power series expansion in the strength of the inducing charge corresponds to the Uehling potential. The authors consider in some detail the third order term and the polarization potential at small distances. They find an entirely negligible contribution from the third order term to the Lamb shift in hydrogen. The shift in energy levels of mu-mesonic atoms from higher than the first order terms are also found to be small. *J. Leite Lopes.*

See also: Moses, p. 216; Nakajima, p. 246; Rubinaw and Wu, p. 257.

Relativity

Krause, H. G. L. Relativistische Raketenmechanik. Astronaut. Acta **2** (1956), fasc. 1, 30–47.

The author extends the relativistic mechanics due to the special theory of relativity to systems in which the rest mass varies with the time (rockets). He examines the theorems of impulse and energy and the law of decrease of mass of an arbitrarily accelerated rocket in a free space devoid of exterior forces. The theorems are given in the coordinate systems of an observer on the earth and of an astronaut moving with the rocket. Finally two special cases are considered: (i) Motion of a rocket with constant proper acceleration and (ii) motion of a rocket with constant rate of mass flow (thrust). *E. Leimanis.*

Mátrai, T. Eine kinematische Deutung des Inertialsystems. Acta Phys. Acad. Sci. Hungar. **5** (1956), 409–423. (Russian summary)

The author develops orthodox special relativity theory from a few primary concepts (e.g. particle, one-dimensional ordering of a particle's events, relative motion of particles) and a number of appeals to observation. By introducing at an early stage the concept of a clock-particle which measures proper time {and which appears to the reviewer to have the same axiomatic status as the primary concepts}, he is able to develop the theory without the introduction either of light signals or of rigid rods. The frequency of the appeals to observation obscures the logical structure of the development, which is nevertheless highly stimulating. *F. A. E. Pirani* (London).

Kar, S. C. Zur Elektromagnetik materieller Körper auf Grundlage der Massformel des Viererraumes. Bull. Calcutta Math. Soc. **47** (1955), 171–189.

The author studies various forms of Maxwell's equations in a moving refracting medium, assuming in the medium an isotropic metric in which the index of refraction appears explicitly. There are many misprints.

F. A. E. Pirani (London).

Marx, G.; und Györgyi, G. Über den Energie-Impuls-Tensor des elektromagnetischen Feldes in Dielektrika. Ann. Physik (6) **16** (1955), 241–256.

This paper is a review article. It summarizes the work of the authors and others discussing the proper form of the energy momentum tensor of the electromagnetic field inside dielectrics. [For the authors' original work see Acta Phys. Acad. Sci. Hungar. 3 (1954), 213–242; MR 16, 775.]

It is shown that only the tensor proposed by Abraham [Rend. Circ. Mat. Palermo **28** (1909), 1–28] has the following properties: 1) It can be derived from the relativistic Lagrangean of the electromagnetic field in the standard fashion by varying the metric tensor. {The authors quote K. F. Novobátzky, Hungar. Acta Phys. **1** (1949), 25–34; MR **11**, 566. However, this result was already obtained by W. Gordon, Ann. Physik. (4) **72** (1923), 421–456.} 2) The law of force and radiation pressure derivable from this tensor is correct for static and stationary fields. 3) The law of force satisfies the conservation laws of energy, momentum and angular momentum, and the center of mass theorems. 4) The momentum flux of the field is equal to the energy flux of the field divided by the square of the light velocity as it is required by Planck's formulation of the law of inertia of energy. 5) From the Abraham tensor and the energy momentum tensor of the medium we can construct the energy momentum tensor of the radiation field inside the medium. (Energy momentum tensor of the radiation field = energy momentum tensor of the electromagnetic field + energy momentum tensor of medium while the field is present — energy momentum tensor of medium in the absence of the field.) The velocity of energy propagation defined by this radiation tensor has the same transformation properties as the velocity of a mass point. 6) The energy density of the Abraham tensor is always positive definite. The quantisation of the field describes photons with a finite and real rest mass. The energy and momentum of a photon form a four vector.

No energy momentum tensor which does not possess these properties can be correct. Consequently the tensor proposed by Minkowski [Math. **68** (1910), 472–525] which violates 1), 3), 4), 6) is incorrect. Abraham's tensor is as yet not in conflict with any known condition.

N. L. Balazs (Chicago, Ill.).

Winterberg, F. Relativistische Zeitdilatation eines künstlichen Satelliten. Astronaut. Acta **2** (1956), fasc. 1, 25–29.

The relativistic treatment of the Doppler effect introduces certain supplementary measurable terms which are closely connected with Einstein's time dilatation. Due to the existence of fields of gravitation the general theory of relativity is applied. Considerations which pertain to the Doppler effect are transferred to the motion of clocks on an artificial satellite and on the surface of the earth. The deviation in the motions of these clocks is measured by the comparison of two quartzclocks, also called atomic clocks, which are regulated by a certain molecular resonance line. Calculations show that a clock on an artificial satellite in comparison with a clock on the earth's surface loses several thousandths of the second in the course of a year. This effect can be reproduced by a clock having a precision of at least 10^{-11} in its motion. Such high precision clocks are the so-called atomic clocks. *E. Leimanis* (Vancouver, B.C.).

Papapetrou, A. Rotverschiebung und Bewegungsgleichungen. Ann. Physik (6) **17** (1956), 214–224.

The author studies the motion of (i) a classical electron, (ii) a Dirac electron, in the field of an atom at rest in a Schwarzschild space-time. He concludes that his considerations yield the usual value for the gravitational redshift. The author's motivation is to deduce the redshift without having to make the hypothesis (*): that the vibrations of an atomic system measure equal intervals of proper time. {However it appears to the reviewer that (*) is an immediate consequence of the covariance of the usual equations of motion of the electron, which the author has to assume in his investigation.}

F. A. E. Pirani (London).

Bonnor, W. B. The instability of the Einstein universe. Monthly Not. Roy. Astr. Soc. **115** (1955), 310–322 (1956).

In a previous paper [Z. Astrophys. **35** (1954), 10–20; MR **17**, 202] the author discussed the junction of two universes across a spacelike 3-space $t=0$. The present paper treats, for the most part, the junction of an Einstein universe with a spherically symmetric disturbance across a timelike 3-space $x_1=r=a$ (the history of a spherical surface), with continuity of

$$g_{ik},\ \partial g_{\mu\nu}/\partial x_1,\ T_k{}^i,\ g_{i\mu}T_k{}^\mu - g_{k\mu}T_i{}^\mu$$

($i, k=1, 2, 3, 4$; $\mu, \nu=2, 3, 4$ [cf. S. O'Brien and J. L. Synge, Communications Dublin Inst. Advanced Studies, Ser. A, no. 9 (1952); MR 14, 913]. For $r>a$ we have the Einstein line element

$$ds^2=-\left(1-\frac{r^2}{R^2}\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2)+dt^2,$$

and for $r<a$ the general spherically symmetric line element

$$ds^2=-e^{\lambda}dr^2-e^{\omega}(d\theta^2+\sin^2\theta\, d\phi^2)+e^{\nu}dt^2,$$

with λ, ω, ν functions of r and t. Examining first the static case, in which λ, ω, ν are independent of t, the author concludes that static inhomogeneities can exist in the Einstein universe, though they may not all be nonsingular. He then proceeds to the general non-static case, making certain approximations (smallness of ν, $\partial\lambda/\partial t$, $\partial\omega/\partial t$), and comes to the conclusion that if there is an equation of state (other than p=const) no non-static approximate models with inhomogeneities are possible, but if there is no such equation, non-static models exist and are in general unstable, though there are special cases in which motion inside $r=a$ can occur without instability, including some in which condensations may begin to form. {The coordinate r in the Einstein line element is identified geometrically by the fact that $4\pi r^2$ is the area of the 2-sphere r=const, t=const. The reviewer is not satisfied that $r=a$ is an appropriate boundary for a non-static disturbance; why should the surface area of the disturbance be constant?} *J. L. Synge* (Dublin).

Zahradníček, J.; und Čížek, A. Perihelverschiebung des Merkur. Publ. Fac. Sci. Univ. Masaryk **1955**, 281–294. (Czech. Russian and German summaries)

Based on the principles of conservation of energy and angular momentum of a planet in the gravitational field of the Sun, with allowance for the dependence of inertial mass on velocity, an expression for the advance of the perihelion of Mercury is derived. The result differs from that obtained by Einstein from general relativity. Numerically, the motion is found to be 34."2 per century as compared with 43" from general relativity. {The latest comprehensive discussion of the observations [G. M. Clemence, Astronomical papers, v. 11, Nautical Almanac Office, Washington, D.C., 1943, pp. 9–221] gives 42."9± 0."3 for the observed effect, in excellent agreement with 42".9 required by general relativity.} *D. Brouwer.*

Maurer-Tison, Françoise. Théorie unitaire et électromagnétisme dans la matière. C. R. Acad. Sci. Paris **242** (1956), 3042–3045.

(I) Let $g_{\lambda\mu}$, $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ be the fundamental tensor of Einstein's unified field theory, its determinant and the determinants of $h_{\lambda\mu}=g_{(\lambda\mu)}$ and $k_{\lambda\mu}=g_{[\lambda\mu]}$ respectively, $\mathfrak{g}<0$, $\mathfrak{h}<0$. Let $g^{\lambda\nu}$, $h^{\lambda\nu}$ be inverse tensors to $g_{\lambda\mu}$, $h_{\lambda\mu}$ and put $l^{\lambda\nu}=g^{(\lambda\nu)}$, $\gamma^{\alpha\beta}=2\mathfrak{h}\mathfrak{g}^{-1}h^{\alpha\beta}-l^{\alpha\beta}$ and consider the Maxwell equations

(1)(a) $\quad \gamma^{\alpha\beta}\nabla_\alpha G_{\beta\mu}+J_\mu=0,\quad$ (b) $\partial_{[\mu}H_{\lambda\nu]}=0$

with

(1)(c) $$G_{\mu\lambda}=l^{\alpha\beta}\gamma_{\mu\beta}H_{\alpha\lambda}.$$

(II) An easy inspection based on the algebraic structure of (1) shows that the problem of Cauchy applied to (1) leads to the characteristic cone defined by $l_{\alpha\beta}$. If in particular

(2) $$2H_{\lambda\mu}=\varepsilon_{\lambda\mu\alpha\beta}g^{\alpha\beta}\sqrt{(-\mathfrak{g})},$$

then (1)(b) is equivalent to Einstein's condition

$$\partial_\lambda\sqrt{(-\mathfrak{g})}g^{[\lambda\nu]}=0.$$

The relationship of the main Einstein's equations

$$R_{(\mu\lambda)}=0,\quad \partial_{[\omega}R_{\mu\lambda]}=0$$

and (1)(a) is not dealt with.

{Remark of the reviewer. Even if $\mathfrak{g}\mathfrak{h}\mathfrak{k}(\mathfrak{g}-2\mathfrak{h})\neq 0$ the tensor $\gamma^{\alpha\beta}$ could be of rank 2 so that its introduction as metric tensor requires additional conditions to be imposed on $g_{\lambda\mu}$.} *V. Hlavatý* (Bloomington, Ind.).

Lichnerowicz, A. Etude des équations du champ de la théorie unitaire d'Einstein. Rend. Sem. Mat. Fis. Milano **25** (1953–54), 121–133 (1955).

The content of this article is substantially the same as J. Rational Mech. Anal. 3 (1954), 487–521 [MR **16**, 408]. *V. Hlavatý* (Bloomington, Ind.).

Mauger, F. E. The strength of field equations. Nuovo Cimento (10) **3** (1956), 1494–1495.

In the theory of the "strength of field equations" Einstein used a number $\mathcal{D}_n$ which the present author rectifies. If T is an arbitrary covariant tensor and

(1) $$x^\nu=x'^\nu+c_{\lambda_1\cdots\lambda_n}{}^\nu x'^{\lambda_1}\cdots x'^{\lambda_n},$$

where the c's are constants, is a coordinate transformation, then at the origin

(2) $$\frac{\partial^n T'}{\partial' x^{\lambda_1}\cdots\partial x'^{\lambda_n}}=\frac{\partial^n T}{\partial x^{\lambda_1}\cdots\partial x^{\lambda_n}}+L_{\lambda_1\cdots\lambda_n},$$

where $L_{\lambda_1\cdots\lambda_n}$ is a bilinear polynomial in c and T. Therefore the lefthand term in (2) depends on B_n numbers c, where

$$B_n=4\binom{4}{n+1}.$$

This number has to be substituted for $\mathcal{D}_n$. {Remark of the reviewer: The actual number of conditions which may be imposed on the lefthand term in (2) depends on the structure of (covariant) derivatives of T.} *V. Hlavatý.*

Eisenhart, Luther P. A unified theory of general relativity of gravitation and electromagnetism. II. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 646–650.

[For part I see same Proc. 42 (1956), 249–251; MR **17**, 1016.] Let

(1) $$\Gamma^\nu_{\lambda\mu}=\left\{{\nu \atop \lambda\mu}\right\}+k_{\lambda\mu}p^\nu,$$

where $\left\{{\nu \atop \lambda\mu}\right\}$ are Christoffel's symbols of the gravitational field tensor $h_{\lambda\mu}$ and $k_{\lambda\mu}$ is the electromagnetic field tensor. Denote by $R_{\lambda\mu}$ $(H_{\lambda\mu})$ the contracted curvature tensor of $\Gamma^\nu_{\lambda\mu}$ $\left(\left\{{\nu \atop \lambda\mu}\right\}\right)$. The author requires with Einstein that

(2) $$R_{(\lambda\mu)}=0.$$

Further conditions imposed by the author are (∇_μ denotes covariant derivative with respect to $\left\{{\nu \atop \lambda\mu}\right\}$)

(3) $\quad \nabla_\mu p_\lambda=p_\lambda p_\alpha k_\mu{}^\alpha+\varepsilon k_{\lambda\mu},\quad p_\alpha p^\alpha=\varepsilon\quad (\varepsilon=0 \text{ or } 1),$

$(4)_1 \quad p^\alpha\nabla_\mu k_{\alpha\lambda}=p_\mu p_\lambda$ const (for $\varepsilon=0$),

$(4)_2 \quad \nabla_\omega k_{\mu\lambda}=2p_\omega p_{[\mu}g_{\lambda]}$ (for $\varepsilon=1$).

For the case $\varepsilon=0$ one obtains easily from (1) (2) $H_{\lambda\mu}=$

$p_\lambda p_\mu$ const, while $\varepsilon=1$ yields by the same token $H_{\lambda\mu}=k_{\lambda\alpha}k_\mu{}^\alpha$ and

$$\nabla^\mu k_{\mu\lambda}=g_\nu(\delta_\lambda^\nu-p_\lambda p^\nu)=0$$

so that we have the case of pure radiation. (Of course in both cases the tensor $H_{\lambda\mu}-\frac{1}{2}H_{\alpha\beta}h^{\alpha\beta}h_{\lambda\mu}$ is divergence free). {Reviewer's remarks: 1) The case (I) in the Einstein theory has been dealt with in Hlavatý, J. Rational Mech. Anal. 2 (1953), 1–52 [MR 14, 505]. 2) The Einstein requirement $\Gamma^\nu_{[\lambda\nu]}$ would eliminate here all fields $k_{\lambda\mu}$ of the first class.} *V. Hlavatý* (Bloomington, Ind.).

Graef Fernández, C. The gravitational forces of Birkhoff in physical space. Univ. Nac. Autonoma Mexico. An. Inst. Fis. 1 (1955), 35–47. (Spanish)

The author takes the equations

$$f_i=\left(\frac{\partial h_{ij}}{\partial x^k}-\frac{\partial h_{jk}}{\partial x^i}\right)\frac{dx^j}{ds}\frac{dx^k}{ds}$$

of Birkhoff's gravitational theory, where

$$f^i=\frac{d^2x^i}{ds^2},\ ds^2=(dx^1)^2-(dx^2)^2-(dx^3)^2-(dx^4)^2,$$

and obtains formulae for the components d^2x/dt^2, d^2y/dt^2, d^2z/dt^2 of the spatial acceleration-vector in terms of the derivatives of the spatial scalar potential $\Phi=h_{11}$, the vector $\mathbf{P}=(h_{12}, h_{13}, h_{14})$, the tensor $T=(h_{\alpha\beta})$ $(\alpha, \beta=2, 3, 4)$, and the components dx/dt, dy/dt, dz/dt of velocity. {Note: The formulae for L_z, Q_z on page 42 are wrongly bracketed: the last term there given for L_z belongs to Q_z, with $\dot{z}^2$ for $\dot{z}$. There are also discrepancies in the signs of some of the terms of Q_x, Q_y, Q_z.} *H. S. Ruse* (Leeds).

See also: Taylor, p. 260; Rayski, p. 260.

Astronomy

Šaraf, Š. G. Theory of the motion of Pluto. Trudy Inst. Teoret. Astr. no. 4 (1955), 5–131. (Russian)

The author gives the analytical development of the first-order perturbations of Pluto from Jupiter, Saturn and Uranus, using the method of Laplace-Newcomb. The disturbing function is developed by Newcomb's method of symbolical operators. Innes' formulae are used to compute the Laplace coefficients and their derivatives. The largest part of the work, including the computations of operators, was performed on the electronic computing machines. The perturbations from Neptune are computed by mean of numerical integration. All observations of Pluto, from 1914 to 1951, are combined to 24 normal places and the corrections in elements are computed using the method of N. Samoilova-Yakhontova. The remaining residuals in α are less than 1."91, the residuals in δ are less than 2."87. An appendix contains the table of Newcomb's operators. The author corrects some numerical errors in Newcomb's work [Astronomical papers, v. 5, Washington, D.C., 1895]. *P. Musen.*

Steffensen, J. F. On the restricted problem of three bodies. Danske Vid. Selsk. Mat.-Fys. Medd. 30 (1956), no. 18, 17 pp.

Let two particles P_1 and P_2 revolve in circular orbits about their common centre of gravitation with constant angular velocity. Consider a third infinitesimal particle P which moves in the plane of motion of P_1 and P_2 in such a way that, while it is subject to the Newtonian attractions of P_1 and P_2, it does not disturb the Keplerian motion of the two finite particles. Expansions for the coordinates of P in powers of the time t are generally considered to be impractical. The author, however, shows that by introduction of certain auxiliary dependent variables, the equations of motion of P can be reduced to a differential system of the second order. This last system permits us to calculate the coefficients of t^r by a set of recurrence formulas which lend themselves easily to modern calculating machines. The method used is closely related to that given in one of the author's recent papers [Acta Math. 95 (1956), 25–37; MR 17, 1141]. Sufficient conditions for the convergence of the resulting series are obtained and a simple illustrative numerical example is given. *E. Leimanis* (Vancouver, B.C.).

Zagar, Francesco. Modelli anisotropi nella cosmologia newtoniana. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 19 (1955), 13–16.

The author continues his previous work [same Rend. (8) 18 (1955), 452–458; MR 17, 1142]. In the previous paper only expanding world-models were considered. In the present work he studies models with contraction as well as expansion. *F. A. E. Pirani* (London).

See also: Kaeppeler, p. 246; v. Hoerner, p. 254; Kagan and Yudin, p. 257; Krause, p. 261; Winterberg, p. 261; Zahradníček and Čížek, p. 262.

Geophysics

Heaps, H. S. The effect of elastic intrusions upon a gravitational stress. Trans. Amer. Geophys. Union 37 (1956), 477–482.

When the elastic properties of intrusions do not differ greatly from those of the surrounding medium their effect is found to be the same as that of a set of body forces. For intrusions sufficiently near the horizontal boundary of a semi-infinite medium they reduce to a single vertical force. By using the results for a concentrated force in an infinite medium the overall effect of many intrusions is investigated. Detailed results are obtained for a single intrusion and it is shown that for a spherical intrusion the maximum shearing stress just outside is independent of the radius. *B. R. Seth.*

Longuet-Higgins, M. S. The refraction of sea waves in shallow water. J. Fluid Mech. 1 (1956), 163–176 (1 plate).

Short-crested sea waves change in character when they are refracted by a shallowing depth of water. Besides a change in mean wavelength and direction there are changes in the mean length of the crests (usually an increase) and in the skewness. The author first considers briefly the refraction of a simple wave-train, and of two long-crested wave-trains which differ slightly either in direction or in wavelength. Next, to describe actual sea waves he assumes that the waves in any given locality are statistically uniform and that they can be described by an energy-density function of the vector wave-number, that each wave-frequency refracts independently, and that reflection from the coast is negligible. These assumptions are not unduly restrictive. Expressions are given for the change in various statistical wave parameters when the

energy density is concentrated in a small region of the wave-number plane; the case of short-crested waves on long-crested swell is also considered. Numerical examples are given. A number of general conclusions are obtained, too many to be given here. The limitations of the analysis are clearly stated and discussed.

{There is much previous work on the non-statistical aspects of this problem [see, e.g., Munk and Arthur, Gravity waves, Nat. Bur. Standards Circular 521, Washington, D.C., 1952, pp. 95–108; MR **14**, 918] and the references given there.} *F. Ursell.*

Fel'zenbaum, A. I. Investigation of connections between wind, distribution of density, level and currents of an inhomogeneous sea. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 958–967. (Russian)

The known general solution of the distribution problem of velocities (currents) and densities which are caused by the action of a known wind on various levels of an inhomogeneous and deep sea of finite extension but arbitrary shape is applied by the author to a particular case when the sea's shape is rectangular and the friction is neglected. *E. Kogbetliantz* (New York, N.Y.).

Thompson, Philip Duncan. A theory of large-scale disturbances in non-geostrophic flow. J. Meteorol. **13** (1956), 251–261.

This paper gives an account of some of the problems involved in the numerical integration of the hydrodynamical equations governing the motion of the atmosphere. By considering a simple linear system the author finds a necessary and sufficient condition for the exclusion of solutions corresponding to gravity waves (sound waves are also excluded, by use of the hydrostatic equilibrium approximation throughout the analysis). This "filtering approximation" is then introduced into the general nonlinear equations and it is shown that this condition is sufficient for the exclusion of gravity waves. The author then describes a possible method of performing the numerical integration of the modified equations, when the initial distribution of pressure is assumed given.

M. H. Rogers (Urbana, Ill.).

Medvedev, S. V. Oscillation of a vertical system under horizontal seismic action. Trudy Geofiz. Inst. no. **36**(163) (1956), 62–79. (Russian)

This problem is studied first in the particular (unreal) case when there is no dissipation of energy, that is, no damping. The proper (eigen-) oscillations for this case are determined. Then the damping is accounted for, the equations of the problem being discussed in Lagrangean coordinates of the system with no damping.

It is found that for the fundamental frequency the effect of damping is so small that in practice it can be neglected. *E. Kogbetliantz* (New York, N.Y.).

★ **Jung, Karl. Figur der Erde.** Handbuch der Physik. Bd. 47, pp. 534–639. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956.

This section of the Handbuch der Physik gives a condensed encyclopedic exposition of the figure of the earth, stressing the physical rather than the geometrical, and the theoretical rather than the empirical, points of view. The author begins by precisely defining the terms geoid, reference ellipsoid, earth ellipsoid, and spheroid (equipotential surface of the earth disregarding irregularities in the earth's crust, but including centrifugal force). Then the theory of the gravitational field of the earth is developed with respect to these surfaces in a concise but comprehensive manner employing the usual method of spherical harmonics. There is a thorough discussion of the movements of the earth's axis due to wandering of the poles and precession. Finally, a very short section is allotted to problems of a practical nature facing geodesists today.

Within this framework the following subjects are covered and theoretical formulations developed: Bruns' theorem concerning the separation of geoid and spheroid; the normal gravity formula in various forms; separation between spheroid and ellipsoid; Clairaut's theorem in extended form; the special spheroids of Bruns, Helmert, McLaurin and Haalck; the variations of gravity, latitude, spheroid separation and spheroid flattening with height; mean formulas such as for mean density and mean gravity on a sphere of volume equal to that of the earth; development of the gravity field of the equipotential ellipsoid according to Jeffreys and Somigliana; Stokes' formula for the determination of geoidal undulations from gravity anomalies; Vening Meinesz's adaptation of Stokes' formula for the determination of deflections of the vertical; Prey's development of the earth's topography in spherical harmonics; determinations of parts of the geoid from observational material; the triaxial ellipsoid; the different types of height defined in geodesy; the gravitational field of the earth's interior; the various components of the polar wandering and their causes; and precession as a function of the gravity field of the sun and the moon.

There are four appendices: the first tabulates normalized spherical harmonics of the first kind as a function of the reduced colatitude; the second gives the coefficients of Prey's development of the earth's topography in spherical harmonics; the third gives an excellent collection of basic formulas covering spherical harmonics, and series expansions and integral formulas related to them; and the fourth lists a varied set of numerical values pertaining to the figure of the earth, including a table of gravity formulas, constants of the earth's rotation, various dimensions computed for the figure of the earth, triaxial ellipsoid determinations, components of the moments of inertia, and level surfaces within the earth as a function of depth.

Certain theoretical aspects are not mentioned or glossed over lightly: geometric theory such as the theory of the geodetic line and the geometry of the ellipsoid; the theory of determination of the earth's size and shape by astronomic and triangulation methods; the determination of the geoid by astrogeodetic deflections of the vertical; and the various isostatic hypotheses. Furthermore, the practical aspects of the subject, although frequently alluded to, are usually mentioned merely in passing: these include topics such as the reduction of topography and gravity measurements. Some limitations, however, are obviously necessary in an article of this size, and the ones the author has chosen would seem to be appropriate in view of the setting of the article within the Handbuch der Physik. As a summary of the theoretical-physical aspects of the figure of the earth, this article presents a collection of facts and formulas which one will find the most convenient to use of all that are in existence today.

There is one unfortunate notational ambiguity which is carried over from other works on geodesy [e.g. Baeschlin, Lehrbuch der Geodäsie, Füssli, Zürich, 1948]. This is the use of 'V' to stand for the potential of the non-rotating

earth (p. 536), and also for $(1+\varepsilon^2 \cos^2 \beta)^{\frac{1}{2}}$, where ε^2 is an ellipsoidal constant and β is the geodetic latitude (p. 569).

B. Chovitz (Washington, D.C.).

Halfin, L. A. The field of a point source in the presence of an oblate or a prolate spheroid. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 657–668. (Russian)

From a classical expansion of potential into a series of spheroidal harmonics, the well-known expressions of its gradient are deduced. They are considered on profiles parallel to axes of spheroids. The spheroids are given with respect to the ground only two particular positions each, in which a principal axis is perpendicular to the ground.

E. Kogbetliantz (New York, N.Y.).

Belluigi, A. Über ein geoelektrisches inverses Problem. Z. Geophys. **21** (1955), 135–151.

The author presents a very interesting and practically important solution of a two-dimensional geoelectric interpretation problem for the case of two layers carrying an uniform steady current, telluric or sent through electrodes in the ground.

The solution is obtained by the method of finite differences which — as the author states — seems to be much more reliable than the Fourier expansion into a series of Hermite polynomials, the use of which is also discussed in the paper.

It is important to mention that the boundary surface separating the two layers (horizontal infinite cylinder) cannot be determined if the survey is based on the study of telluric currents. *E. Kogbetliantz.*

Simonenko, T. N. On the computation of the Z_a values from the measured values of ΔT. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 704–707. (Russian)

An aeromagnetic survey yields a picture of the total magnetic intensity and its interpretation can be singularly facilitated if it is transformed by computation into a map of vertical component Z of the total magnetic field. A very efficient numerical method of this transformation is discussed in this paper. Its simplicity is due to an essential limitation not mentioned in the title: only the two-dimensional case is considered which means that the double integration necessary in the general case is reduced to a simple integral taken along a rectilinear profile perpendicular to the family of parallel isolines representing the magnetic anomaly. This integral is evaluated as Cauchy's principal value of a divergent integral which is perfectly correct. As a check serves a numerical example in which the vertical component was known as a result of an ordinary magnetic survey. The computed curve coincides well with the directly observed curve of Z. The case considered in this example was a favourable one: the magnetic anomaly was 75 klm. long and 30 klm. wide so that the isolines were in fact parallel, the profile being laid over the central part of the anomaly.

E. Kogbetliantz (New York, N.Y.).

See also: de Castro Brzezicki, p. 205; Press and Tukey, p. 245; Dolukhanov, p. 257.

OTHER APPLICATIONS

Games, Economics

Jongmans, F. Le problème duopoliste. Bull. Soc. Roy. Sci. Liège **24** (1955), 326–365.

The author reconsiders the well-known duopoly problem: Two producers choose outputs x_1, x_2, respectively; the price is then given by the demand function $f(x_1+x_2)$, and the costs to the two producers are $F_1(x_1)$, $F_2(x_2)$, respectively. Since each producer wishes to maximize his profit, how should they choose x_1 and x_2? Let r_i be the profit of producer i; then $r_i=x_if(x_1+x_2)-F_i(x_i)$ $(i=1, 2)$. The author argues that equilibrium cannot be achieved except at a "dominant" point, that is, at a pair (x_1, x_2) such that any change from it must decrease one of r_1, r_2. Among these points he distinguishes those which are stable according to the following (admittedly imprecise) criterion: for every possible change in one output, say x_1, which increases r_1, there exists a corresponding change in x_2 which reduces r_1 below its original level and more or less compensates for the initial reduction in r_2. The author makes the following assumptions: $f(x)$ is continuous and decreasing over the closed interval $[0, X]$; $f'(x)<0$ and $2f'(x)+xf''(x)\leqq 0$ over the same interval; $f(x)=0$ for $x\geqq X$; $F_i(x_i)$ is twice continuously differentiable and it and its first two derivatives are positive for positive values of the argument. Under these assumptions, he shows that if μ is the point which maximizes r_1+r_2, then there is a high degree of stability in his sense for points (x_1, x_2) near to μ. On the other hand, dominant points close to those which maximize r_1 or r_2 will tend to be unstable. The analysis, as suggested above, tends to be suggestive rather than conclusive. The mathematical argument is interwoven with a great many penetrating remarks about the possible strategies of duopolists.

K. J. Arrow (Stanford, Calif.).

Isard, Walter. Some remarks on the marginal rate of substitution between distance inputs and location theory. Metroecon. **5** (1953), 11–21.

Imagine a uniform plane, with resources, demand, and identical producers uniformly distributed over it. The author remarks that if locational patterns and market areas are constituted to maximize the standard sort of "social surplus" inclusive of transportation costs [Isard, Econometrica **20** (1952), 406–430; MR **16**, 500], then the resulting pattern of market areas must consist of regular hexagons packing the plane. *R. Solow.*

★**Klein, Lawrence R. A textbook of econometrics.** Row, Peterson and Company, Evanston, Illinois; White Plains, New York, 1956. ix+355 pp.

This is the second unchanged printing of the well-known textbook on econometrics, originally published in 1953. The book centers around the theory of estimation of economic models. In Dr. Klein's exposition, the estimation theory gets a very definite flavour of the estimation methods developed within the Cowles Commission (University of Chicago).

The book is intended to be self-contained for students who have had two years of college mathematics. The statistical groundwork that is essential to a comprehension of econometrics is outlined in chapter two. There is also an appendix to the book on the algebra of determinants and matrices, which the student can consult in connection with certain sections. Thus prepared, Klein after an opening chapter on the link between economic

theory and econometrics proceeds to a systematic description of estimation methods of aggregative models. The material is to a large extent presented by means of practical examples, many of which are taken from models that Klein has used in his previous works. This numerous use of illustrations to the theoretical methods seems to be particularly successful. The maximum likelihood method and the limited information method for the estimation of the parameters of econometric models are described in detail. There is then a valuable chapter on computational design of least squares, maximum likelihood and limited information estimation techniques. The rest of the book is devoted to methods of sector analysis and special problems. The theory of estimation of sector models is developed as a theory of "grafting", i.e. a shortcut closing of the system to an over-all aggregative master model. The treatment of estimation of cross section data gives some very interesting examples from Klein's experience of the savings surveys conducted at the Survey Research Center, University of Michigan.

While this book is mainly written in the spirit of identification and maximum likelihood theory, it is important to point out that Dr Klein never takes such a vigorous stand on the question maximum likelihood versus least squares as have many other authors. It is also interesting to note that Klein in his later works, published after this textbook, has come to be known for his very successful analyses of household savings data, using solely ordinary least squares.

One would have wished that the author had used this opportunity to correct some printing errors of the first printing (e.g. a disturbing manuscript error on pp. 220–221).

S. A. O. Thore (Stockholm).

Nikaidô, Hukukane. On the classical multilateral exchange problem. Metroecon. 8 (1956), 135–145.

The exchange model deals with m individuals and k commodities. Let $U_i(X_i)$ be the utility function for the ith individual, and let his initial holdings be a vector A_i. For any given set of prices, he is assumed to choose X_i to maximize $U_i(X_i)$ subject to the condition $PX_i=PA_i$; the problem is to prove the existence of a price vector P such that $\sum_{i=1}^m X_i=\sum_{i=1}^m A_i$ (demand equals supply). This is proved under the assumptions that the functions U_i are quasi-concave and strictly increasing. The technique is to map prices into sets of prices and use Kakutani's fixed-point theorem. More specifically, define $\phi_i(P)$ as the set of X_i's which maximize $U_i(X)$ subject to $PX=PA_i$, and $\phi(P)$ as the direct sum of the sets $\phi_i(P)$; $\phi(P)$ maps the P-space into subsets of the X-space. For any X, let $\theta(X)$ be the set of vectors P which maximize $P(X-\sum_{i=1}^m A_i)$ over the fundamental simplex. The two mappings together map the fundamental simplex in P-space into subsets of itself; the combined mapping can be shown to satisfy the conditions of Kakutani's theorem [Duke Math. J. 8 (1941), 457–459; MR 3, 60], and it is easy to verify that a fixed point of the point-to-set mapping solves the original problem. Some generalizations to foreign trade models are given.

The paper extends some earlier work of L. McKenzie [Econometrica 22 (1954), 147–161]. It overlaps but is independent of existence proofs for competitive equilibrium by Arrow and Debreu [ibid. 22 (1954), 265–290; MR 17, 985] and Gale [Math. Scand. 3 (1955), 155–169; MR 17, 985].

K. J. Arrow (Stanford, Calif.).

Bronfenbrenner, M. An elasticity of inflation. Metroecon. 8 (1956), 107–117.

The author observes that the elasticity of inflation (relative change of price with respect to money income) is given by the following formula:

$$I=\varepsilon/(e+\eta),$$

where ε is the money income elasticity of demand, e is the money price elasticity of supply, and η the money income elasticity of demand (considered as positive). This relation is discussed in both micro-economic and macro-economic contexts. In the former it provides some basis for theorizing about relative price movements, in the latter for interpreting the different theoretical approaches (quantity theory, Keynesian, price-flexibility). *K. J. Arrow.*

Kemeny, John G.; Morgenstern, Oskar; and Thompson, Gerald L. A generalization of the von Neumann model of an expanding economy. Econometrica 24 (1956), 115–135.

Following the work of von Neumann [Rev. Econ. Studies 13 (1945–46), 1–9], the authors consider a linear economy operating over time. Let x_i be the intensity of a process, a_{ij} the amount of commodity j used in one unit of process i, b_{ij} the amount of commodity j produced by one unit of process i, y_j the price of commodity j, and A and B the (non-negative) matrices (a_{ij}) and (b_{ij}). Production takes one time unit. The economy is expanding at a rate α and paying an interest rate β. The equilibrium conditions of von Neumann require that the expansion be feasible (requiring at any time no more inputs than outputs), that processes do not have positive profit (taking account of interest costs), that inefficient process not be used, and that overproduced commodities have zero prices. Von Neumann then proved that if $A+B$ is a positive matrix (every entry positive), there is a unique $\alpha=\beta$ for which the equilibrium conditions are satisfied. This condition is economically too strong. The present authors impose an additional condition, that $xBy>0$ (value of output is positive), and call such solutions economic. They weaken von Neumann's condition to require every process have at least one input and that every commodity is produced by at least one process. It remains true that for any equilibrium situation, $\alpha=\beta$. From this it follows that allowable rates of expansion must be such that the value of the game $A-\alpha B$ must be zero, and the solutions to this game are the equilibrium x and y; however only a finite number of such α's yield economic solutions. There exists at least one such α. The different α's correspond to different sub-economies of the original economic system. The largest technically possible expansion rate (the largest α for which $xB\geqq\alpha xA$) is largest allowable equilibrium expansion rate; the smallest interest rate which yields a profitless economy ($By\leqq\beta Ay$) is the smallest allowable equilibrium expansion rate.

There are a number of additional problems considered, of which the economically most interesting is the introduction of final demand (other than for additional production) assumed to increase at the same rate as output. *K. J. Arrow* (Stanford, Calif.).

Holt, Charles C.; Modigliani, Franco; and Muth, John F. Derivation of a linear decision rule for production and employment. Management Sci. 2 (1956), 159–177.

In general, the problem of determining optimal production schedules involves non-analytic functions such as max $(x, 0)$. In their first paper [Management Sci. 2 (1955),

1–30] Holt, Modigliani, and Simon showed that quadratic approximations lead to quite manageable problems. In this paper, the authors continue along these lines, presenting the numerical results in detail. *R. Bellman.*

Orden, Alex. The transhipment problem. Management Sci. 2 (1956), 276–285.

The usual transportation problem specifies a set of origins with specified amounts shipped, a set of destinations with specified amounts to be received at each, and a cost of shipping from each origin to each destination. The present author generalizes this problem by permitting transhipment, i.e., goods going from an origin to a destination may go through another port (either origin or destination) or several such ports instead of going directly. This transhipment problem is reduced to a transportation problem. The latter can be written in the form of minimizing $\sum_i \sum_j c_{ij}x_{ij}$ subject to the constraints $\sum_i x_{ij}=b_i$, $\sum_j x_{ij}=a_i$, $x_{ij}\geq 0$, where i varies over the origins and j over the destinations. The transhipment problem requires treating every port as both origin and destination. Let g_i be the amount required to be shipped by the ith port (negative for a destination), and let $c_{ii}=0$. Then choose numbers a_i, b_i so that $g_i=a_i-b_i$; for a_i, b_i sufficiently large, the solution to the resulting transportation problem will have $x_{ii}>0$. The x_{ij}'s of that solution $(i\neq j)$ will be the solution to the transhipment problem. The number of links between ports will in nondegenerate cases be the same as the number of links in the corresponding transportation problem.

The transhipment problem differs from the corresponding transportation problem because the direct link from i to j may not be optimal (in the sense of minimum cost). The optimal route could be found by a transhipment problem with $g_i=1$, $g_j=-1$, $g_k=0$ for $k\neq i$, j. The resulting minimum costs could be substituted for the original cost matrix in future transportation problems with the same network. *K. J. Arrow.*

Predetti, Aldo. Come si risolve un problema di programmazione lineare mediante il metodo del potenziale logaritmico. Statistica, Bologna 16 (1956), 189–211.

Expository paper. A practical linear programming problem is solved by using the logarithmic potential method, due to R. Frisch ["Methods of solving linear programming problems", Seminario Internazionale sull' Analisi dell'Input-Output, Varenna, June-July, 1954].

A. G. Azpeitia (Providence, R.I.).

Markowitz, Harry. The optimization of a quadratic function subject to linear constraints. Naval Res. Logist. Quart. 3 (1956), 111–133.

Let V be a positive semi-definite quadratic form in $X_1, \cdots, X_n$, E a linear form in the same variables, where the variables are non-negative and constrained to satisfy certain linear equations and inequalities. The author is primarily concerned with determining all efficient combinations of E and V, i.e., the points which minimize V for given E and maximize E for given V. The essential point of his algorithm is an application of the theorem of Kuhn and Tucker [Proc. 2nd Berkeley Symposium Math. Statist. Prob. 1950, Univ. of California Press, 1951, pp. 481–492; MR 13, 855] on nonlinear programming. Let λ be a vector of Lagrange multipliers corresponding to the linear constraints and let λ_E be the Lagrange multiplier corresponding to a constraint $E=E_0$. Consider the problem of minimizing V subject to the given linear constraints and to the constraint $E=E_0$. Once the positive variables and the effective constraints are specified, the Lagrangean equations can be written down. As E_0 varies, these equations define X and λ as linear functions of λ_E. They remain valid as defining efficient solutions until one or more of the inequalities in the Kuhn-Tucker conditions ceases to hold. At that point, the set of positive variables or of effective constraints changes, and there is a new critical line. By continuing in this way a finite number of critical line segments are found. The X-projection of these lines in (X, λ)-space are the X-values of the efficient points. The mapping of these points into (E, V)-space is a piecewise parabolic set. The treatment by the author is very complete; he shows how to start the process sketched above and how to treat some degeneracy problems which can arise in changing from one critical line to another. When $\lambda_E=0$ in the above procedure, we have the minimum of V subject to the given linear constraints. Hence, the above procedure also supplies a method analogous to the simplex method of linear programming, of minimizing a quadratic function subject to linear inequalietis. Several variations of this method are sketched. *K. J. Arrow* (Stanford, Calif.).

See also: Jensen, p. 241; Koyck, p. 244; Theil, p. 244.

Biology and Sociology

Rashevsky, N. The geometrization of biology: a correction. Bull. Math. Biophys. 18 (1956), 233–235.

The correction refers to the article listed in MR 17, 761.

Gulliksen, Harold. Measurement of subjective values. Psychometrika 21 (1956), 229–244.

See also: Lah, p. 241; Anderson, p. 243; Harris, p. 257.

HISTORY, BIOGRAPHY

★ **Wilson, Curtis. William Heytesbury: Medieval logic and the rise of mathematical physics.** University of Wisconsin Press, Madison, 1956. xii+219 pp. $4.00.

This book is No. 3 of the University of Wisconsin Press' "Publications in Medieval Science", the two previous books being "The medieval science of weights" [1952, ed. E. A. Moody and M. Clagett] and "Thomas of Bradwardine: His Tractatus de Proportionibus" [1955, ed. H. L. Crosby, Jr.]. The present book is a careful investigation of the "Regule solvendi sophismata", by William of Heytesbury, a colleague of Bradwardine and Suiseth at Merton College, Oxford, in the fourteenth century. The "Regule" may have been composed in 1335 and are extant in several manuscripts and in three editions, all incunabula. They are important for their logical and mathematical discussion of certain cinematical problems, not tested empirically, but taken entirely as thought experiments. Heytesbury shows appreciation of what we now call the limit concept for the analysis of instantaneous motion and velocity, recognizes infinite aggregates in the analysis of the continuum, and has understanding of uniformly accelerated and decelerated motions. In his

discussion "de maximo and minimo" we find certain concepts affiliated to modern ideas on inferior and superior extrema of a numerical aggregate. This study is also helpful for the understanding of the scholastic method of sophismata. Comparison is made with work on the same subject by Suiseth, Duns Scotus and other scholastic writers. *D. J. Struik* (Cambridge, Mass.).

★**van der Waerden, B. L. Erwachende Wissenschaft. Ägyptische, babylonische und griechische Mathematik.** Birkhäuser Verlag, Basel und Stuttgart, 1956. 488 pp. 37.50 francs suisses.

A translation, with modifications and numerous additional illustrations suggested for the most part by the English edition [MR **16**, 1], of the author's "Ontwakende wetenschap" [Noordhoff, Groningen 1950, MR **12**, 381]. The translation into German is by Helga Habicht-van der Waerden.

Stamatis, Evangelos. Über die mathematische Stelle des Theaetetus von Platon. Prakt. Akad. Athēnōn **31** (1956), 10–16. (Greek. German summary)

★**Dijksterhuis, E. J. The Arenarius of Archimedes.** E. J. Brill, Leiden, 1956. 24 pp. 1.90 guilders.

This is the standard edition of the Textus Minores without apparatus criticus. The glossary is Greek-Dutch-English, with a few introductory remarks about the Doric dialect used by Archimedes.

Jamitzer, Wentzeln. Perspectiva corporum regularium. Elem. Math. **11** (1956), 97–100. (German)

Three plates from Jamitzer's book of 1568 are reproduced with brief comment. The plates show some elegantly shaded drawings of solid models of polyhedra, including the icosahedron, dodecahedron, icosidodecahedron, and rhombicosidodecahedron. One of the more complicated figures bears a remarkable resemblance to the great dodecahedron $\{5, 5/2\}$ of Poinsot [J. Ecole Polytechn. **4** (1810), no. 10, 16–48]; but careful examination reveals discrepancies which make it doubtful that Jamitzer actually envisaged a regular polyhedron with interpenetrating pentagonal faces. This doubt is reinforced by the remaining figures, whose beauty is evidently more artistic than mathematical. *H. S. M. Coxeter.*

Hofmann, Jos. E. Über Jakob Bernoullis Beiträge zur Infinitesimalmathematik. Enseignement Math. (2) **2** (1956), 61–171.

Crommelin, C. A. Sur l'attitude de Huygens envers le calcul infinitésimal et sur deux courbes interessantes du même savant. Simon Stevin **31** (1956), 5–18.

Berriman, A. E. The Babylonian quadratic equation. Math. Gaz. **40** (1956), 185–192.

In 13 typical examples drawn from Old-Babylonian mathematical texts written more than 3500 years ago, symbols are used to show at a glance the significance of each successive instruction found in the texts in reaching the solutions of some quadratic equations. The texts in question were edited by O. Neugebauer, Mathematische Keilschrift-Texte [Springer, Berlin, 1935] and O. Neugebauer and A. Sachs, Mathematical cuneiform texts [American Oriental Society and the American Schools of Oriental Research, New Haven, Conn., 1945; MR **8**, 1]. *A. Sachs* (Providence, R.I.).

★**Bellivier, André. Henri Poincaré, ou la vocation souveraine. Vocations, IV.** NRF Gallimard, Paris, 1956. 247 pp. 620 francs.

van Veen, S. C. In memoriam Prof. Dr. J. Haantjes. Simon Stevin **31** (1956), 3–4 (1 plate). (Dutch)

★**Krylov, A. N. [Крылов, А.Н.] Собрание трудов. XII, часть первая. [Collected works. XII, part one. Various works.]** Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. 345 pp. (1 plate). 20.95 rubles.

As the subtitle indicates, this volume contains papers and notes on diverse topics. Some notes (studies for proposed projects, etc.) are published here for the first time.

★**Volterra, Vito. Opere matematiche. Memorie e note. Vol. II. 1893-1899.** Accademia Nazionale dei Lincei, Roma, 1956. iii+626 pp. 8000 lire.

For vol. I see MR **16**, 2. This volume contains Volterra's papers from the years 1893–1899.

★**Levi-Civita, Tullio. Opere matematiche. Memorie e note. Vol. II. 1901-1907.** Pubblicate a cura dell'Accademia Nazionale dei Lincei. Nicola Zanichelli Editore, Bologna, 1956. iii+635 pp. 9000 lire.

For vol. I see MR **16**, 1. This volume contains Levi-Civita's papers from the years 1901–1907.

See also: Massera, p. 227.

MISCELLANEOUS

★**Modern mathematics for the engineer. Edited by Edwin F. Beckenbach.** McGraw-Hill Book Co., Inc., New York-Toronto-London, 1956. xx+514 pp. $7.50.

The book has three parts: mathematical models, probabilistic problems, and computational considerations. Part One has seven chapters: linear and non-linear oscillations, by S. Lefschetz; stability theory, by R. Bellman; exterior ballistics, by J. W. Green; calculus of variations, by M. R. Hestenes; hyperbolic equations, by R. Courant; elliptic equations, by M. M. Schiffer; elastostatics, by I. S. Sokolnikoff. Part Two has five chapters: prediction, by N. Wiener; games, by F. Bohnenblust; operations research, by G. W. King; dynamic programming, by R. Bellman; Monte Carlo methods, by G. W. Brown. Part Three has seven chapters; matrices, by L. A. Pipes; functional transformations, by J. L. Barnes; conformal mapping, by E. S. Beckenbach; nonlinear methods, by C. B. Morrey, Jr.; relaxation methods, by G. E. Forsythe; methods of steep descent, by C. B. Tompkins; high-speed computing devices, by D. H. Lehmer. Each chapter was originally given as a lecture in an extension course at the University of California, Los Angeles, and also at the Corona Laboratories of the National Bureau of Standards.

★**Weyl, F. J. Report on a survey of training and research in applied mathematics in the United States.** Conducted by the National Research Council under contract with the National Science Foundation and published by the Society for Industrial and Applied Mathematics, Philadelphia, 1956. vi+58 pp. $2.00.